Łukasiewicz, Jaśkowski and natural deduction[*]

(Curry-Howard for classical logic)

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§1 Sometime during 1926, while still an undergraduate mathematics student at the Warsaw University, Stanisław Jaśkowski (1906–1965) presented, in the local (i.e., Warsaw) Logic Seminar of his teacher, Jan Łukasiewicz (1878–1956), a 'natural deduction' formulation of classical (two-valued) propositional (including propositional quantifiers), first and second-order logic.

Let us pause, first, on *historico-bibliographical* details. Apparently, the work was done at the instigation of Jan Łukasiewicz[1]. As to terminology, the phrase 'natural deduction' (German: 'natürliche Schliessen'), still in common use in logic today, appeared in print first in Gentzen (1934–1935), although the idea was already clear in the motivation of Jaśkowski's research: he meant, on the authority of his teacher, to design a logic of rules (a 'Regellogik', so to speak), close to the 'natural' mathematical reasoning, as opposed to the 'Satzlogik' of Łukasiewicz himself, i.e., an axiomatic presentation – as in the lectures of Łukasiewicz (1929) –, in the footsteps of Frege (1879), Peirce (1885), Russell (1906), and Whitehead & Russell's 'Principia Mathematica' (1910–1913).[2]

[*]This is a revision of a draft dated Nijmegen, October 9, 2015, based on previous work. In view of my title, I will adopt, throughout in these notes, the Łukasiewicz notational habits.

The phrase 'deduction theory' [Polish: 'teorja dedukcji'] was, initially, Łukasiewicz's own term for 'propositional logic', including, possibly, propositional and / or first- and second-order quantifiers). Cf. the introductory lines and other occasional side-remarks appearing in Jaśkowski (1934).[3]

As to publication matters, Jaśkowski's results were promptly announced, as Jaśkowski (1927), at the First Congress of the Polish Mathematicians, held in Lvov, September 7–10, 1927. See, e.g., the Congress [PPZM] Proceedings (1929) and, specifically, the references of Lindenbaum (1927). Due to circumstances unknown to me (as well as to other, better informed people, apparently), the final paper appeared actually in print only eight years later (in a projected logic collection edited by Jan Łukasiewicz himself, later to become an international logic journal), as Jaśkowski (1934), more or less simultaneously with Gerhard Gentzen's Göttingen *Inauguraldissertation*, Gentzen (1934–1935).

Before going into the proper details of the subject announced in the title, a few more technical and historical remarks on the material available in print – or otherwise – to Jaśkowski, around 1926, are in order.

Modern logic – also called 'mathematical' or 'symbolic' – was (re-) born by the end of the XIX-th century (around 1879, in print, with an entertaining sequel, in 1893), in two instalments, authored by Gottlob Frege, viz. Frege (1879) [BS] and Frege (1893) [GGA:1]. There was an intermediate episode, due to Charles S. Peirce, Peirce (1885) – that Frege ignored –, equally worth noting, which, although sketchy, was, in some respects, conceptually superior to Frege's BS-account. Both Frege and Peirce had a venerable predecessor, more than twenty one centuries before they were born, in the work of Chrysippus of Sol[o]i (this was an obscure place in *Cylicia Campestris*, nowadays in modern Turkey), a Phoenician emigrant to Athens, the father of Stoic logic and the grand-father of [classical] logic *tout court*. The latter (historical) fact was first noted by Jan Łukasiewicz, sometime during the early 1920.[4]

A less known (historical) fact is that Frege came out with two 'logics', not with a single one: a *Satzlogik* (1879), and a *Regellogik* (1893). Both were 'axiomatic', by modern standards. The latter one was meant to be closer to actual 'mathematical thinking' (a kind of formal counterpart of 'natural' deduction, as occurring in mathematical texts), and was vastly anticipating, among other things, Gentzen (1934–1935), for instance.

The other relevant (historical) fact is that neither Gentzen, nor any other *Göttinger* – David Hilbert (1862–1943) or Paul Hertz (1880–1940), for that matter[5] – had ever read Frege's GGA (1893) [sic].

In this matter, I cannot, however, speak for the Lvov-Warsaw Poles – those active in logic before cca 1935 – because we lack the right kind of (historical) documentation. Both Łukasiewicz (1929) and Jaśkowski (1934) mention only Frege's axiomatic system of BS, while, curiously enough, the young Alfred Tajtelbaum [aka Tarski] (1901–1983) does not refer to Frege's 'logics' at all.[6]

On the other hand, the main trouble with Frege (1879, 1893) was in the fact he did not recognise the *general* concept of a 'rule of inference'. Specifically, with [material] implication, [classical] negation and the [classical] universal quantifier as primitives, he only acknowledged 'flat' rules (more or less like the algebraic operations), of the form:

$$(\flat) \vdash \alpha_1, ..., \vdash \alpha_n \Rightarrow \vdash \beta,$$

rejecting, implicitly, the (old-fashioned) idea of *entailment* (= finite sequence of propositions, with exactly one being tagged *qua* 'conclusion'):

$$(\vdash)\ \alpha_1, ..., \alpha_n \vdash \beta$$

as a legitimate – and otherwise essential – logic concept.[7]

The young Bertrand Russell (1872–1970) – the only (more or less competent) person who did actually read Frege (1893) in the epoch[8] – was less interested in such absconse distinctions[9], so he missed the point, as well, and sticked to axiomatics, in the shadow of Frege (1879) and Peirce (1885)[10].

As another aside, I was, so far, unable to date exactly the event as such, in the moderns, viz. the identification of the concept of a *general rule of inference*.[11] The earliest date I am able to quote is 1921, when Alfred Tarski (Leśniewski's only PhD student) noticed the so-called 'Deduction Theorem' (DT)[12], i.e., the implication-introduction rule of Gentzen (1934–1935), an obvious case of 'non-flat' inferential rule, with an entailement (including 'assumptions') as a premiss:

$$(\mathrm{DT})\ \alpha \vdash \beta \Rightarrow \vdash \mathrm{C}\alpha\beta,$$

Certainly, Łukasiewicz was aware of such details, if not in 1921, at least sometime before 1926, when he assigned his [very] young student (Jaśkowski) the home-work leading to Jaśkowski (1927, 1934). Anyway, the implication int-elim rules (the 'Deduction Theorem' and the famous *modus ponens* /

detachment rule) appear explicitly in Jaśkowski's early home-work, and so, *a fortiori*, in Łukasiewicz's Warsaw Seminar, sometime during 1926.[13]

Whence a question: 'Why hasn't Jan Łukasiewicz solved the problem [the one assigned to young Jaśkowski] himself – sometime before 1926 – and presented the outcome is his famous lectures Łukasiewicz (1929)?' Because he had at hand the (conceptual and technical) means to do it, anyway. Which is *what I mean to show next*. In order to do this, in proper terms, I need a *conceptual revision* of the received views on proving (in logic) and some apropriate notation and terminology.

§2 *Rules of inference as witness-operators.* A typical case of the 'flat' rule ($\flat$) above is the *modus ponens* or the *detachment* rule, in logics with implication [here, C], either primitive or defined:

$$(\triangleright)\ \vdash \mathrm{C}\alpha\beta, \vdash \alpha \Rightarrow \vdash \beta,$$

Now, in axiomatic presentations of a given logic (classical, two-valued logic, for instance), the premisses α_i $(0 < i < n+1)$ of a 'flat' rule of the form ($\flat$) are taken to hold 'unconditionally', without further assumptions, they are provable formulas (expressing propositions / propositional schemes), 'theorems' or 'theses' (in the jargon of the early Polish 'school'); alternatively, they are, semantically, *true* (or else two-valued '*tautologies*', in the classical case).

In fact, any particular axiomatics amounts to an *inductive definition* of the predicate 'provable' (expressed notationally by $\vdash$) applying to formulas (expressing propositions or propositional schemes): the axioms are paradigmatically provable (the basis of the induction), while any (primitive) 'flat' rule of inference carries this property – *provability* – from premisses to conclusion (inductive step).

As long as we have only primitive rules of the form ($\flat$) around, 'proving axiomatically' amounts to a piece of *algebraic notation*: a 'flat' *primitive rule of inference* with n premisses $(n > 0)$ looks like a usual algebraic n-ary operation[14], whereupon a *derivable rule of inference* is just an *explicit definition* of an operation in terms of 'primitive' operations (here, axioms and primitive rules of inference).

'Operations on what?', one might wonder. A first – approximate – answer could be: 'On formulas.'.[15] A slightly better one would amount to an additional piece of formalism, to be justified, intuitively, as follows:

Proving something – a proposition expressed by formula α, say – amounts to *providing a reason – or 'grounds' – for* α, or else, like in court, to displaying *a witness* a for α. Formal notation: $\vdash \mathrm{a} : \alpha$.

With this minimal formal equipment, in axiomatic presentations, the axioms are to be witnessed by *primitive constants* (possibly parametric, in the case of axioms schemes), whence 'witnessing' a 'flat' rule of the form ($\flat$) would amount to providing an operation (operator) $\flat$ and a piece of explicit formal notation $\flat(\mathrm{a}_1, ..., \mathrm{a}_n)$, such that

$$[\flat] \vdash \mathrm{a}_1 : \alpha_1, ..., \vdash \mathrm{a}_n : \alpha_n \Rightarrow \vdash \flat(\mathrm{a}_1, ..., \mathrm{a}_n) : \beta;$$

so, in particular, 'witnessing' a 'flat' rule like *modus ponens*, for instance, would consists of using a binary operation $\triangleright$, say, to the effect that

$$[\triangleright] \vdash \mathrm{f} : \mathrm{C}\alpha\beta, \vdash \mathrm{a} : \alpha \Rightarrow \vdash (\mathrm{f} \triangleright \mathrm{a}) : \beta.$$

Summing up, a 'flat' rule of inference is just an *algebraic operation*, in this view. Note, however, that, as long as we do not *define explicitly* the 'operations' $\flat$, we have only a *witness notation*, at most. In other words, in order to have a *witness theory* – as a formal counterpart of (axiomatic) proving – we must be able to *characterise the witness operations* first. Usually, we can do this, like in algebra, by *equational conditions*, expressing *witness-* or *proof-isomorphisms*.

The general case is obtained from the 'flat' case by 'parametrisation' so to speak, where the parameters are finite (possibly empty) sequences of formulas (expressing propositions, resp. propositional schemes) $\Gamma_i := [\beta_{i,1}, ..., \beta_{i,m_i}]$ $(0 < i < n+1,\ m_i \geq 0)$, resp. $\Gamma := [\beta_1, ..., \beta_m]$ $(m \geq 0)$, called 'assumption contexts' (alternatively: *witness-contexts* or *proof-contexts*). Every premiss of a general rule is thus an entailment of the form $\Gamma_i \vdash \alpha_i$ $(0 < i < n+1)$, while the rule has a conclusion of the form $\Gamma \vdash \beta$, i.e., one has

$$(\flat\vdash)\ \Gamma_1 \vdash \alpha_1, ..., \Gamma_n \vdash \alpha_n \Rightarrow \Gamma \vdash \beta,$$

with 'witnessed' counterpart of the form

$$[\flat\vdash]\ \hat{\Gamma}_1 \vdash \mathrm{a}_1 : \alpha_1, ..., \hat{\Gamma}_n \vdash \mathrm{a}_n : \alpha_n \Rightarrow \hat{\Gamma} \vdash \mathrm{b} : \beta,$$

where $\hat{\Gamma}_i := [\mathrm{x}_{i,1} : \beta_{i,1}, ..., \mathrm{x}_{i,m} : \beta_{i,m_i}]$, the 'witnessed' counterpart of Γ_i (and analogously for $\hat{\Gamma}$ and Γ) contains 'decorated – or typed – witness-variables', allowing us to manipulate the witness-contexts.[16]

In particular, in the limit case, a null-premiss rule of inference is just an entailment (considered valid). Examples in point:

[id] $\alpha \vdash \alpha$,

or more generally,

[prj] $\Gamma \vdash \alpha_i$, for $\Gamma \equiv [\alpha_1, ..., \alpha_n]$, $(0 < i < n + 1)$,

[$\triangleright$] $\mathrm{C}\alpha\beta$, $\alpha \vdash \beta$ (*modus ponens*, viewed as a valid entailment),

etc., and analogously for the witnessed variants:

[id$\vdash$] $\mathrm{x} : \alpha \vdash \mathrm{x} : \alpha$,

resp.

[prj$\vdash$] $\hat{\Gamma} \vdash \mathrm{x}_i : \alpha$, for $\hat{\Gamma} \equiv [\mathrm{x}_1 : \alpha_1, ..., \mathrm{x}_n : \alpha_n]$, $(0 < i < n + 1)$,

[$\triangleright\vdash$] $\mathrm{z} : \mathrm{C}\alpha\beta$, $\mathrm{x}\!: \alpha \vdash (\mathrm{z} \triangleright \mathrm{x}) : \beta$.

Here, in [$\flat\vdash$], the witness b, appearing in the conclusion, must be of the form $\flat(\natural_1(\mathrm{a}_1),...,\natural_n(\mathrm{a}_n))$, where the prefixes $\natural_i$ $(0 < i < n+1)$ are either empty (*nil*) or specific variable-binding operations, called 'abstraction operators', acting on finite sequences $\vec{\mathbf{x}}_i \equiv [\mathrm{x}_{i,1}, ..., \mathrm{x}_{i,m_i}]$, $(m_i > 0)$, of *pairwise distinct* witness-*variables*) and the associated 'body' a_i. In each case, a witness-variable is decorated (or 'typed') by an associated formula thereby witnessed 'hypothetically'.

Of course, if every $\natural$-prefix is empty, we have a 'flat' rule, the 'degenerated' case. E.g., in particular, the most general forms of *modus ponens*, viewed as a rule of inference, should be

[$\triangleright\vdash\otimes$] $\hat{\Gamma} \vdash \mathrm{f} : \mathrm{C}\alpha\beta$, $\hat{\Gamma} \vdash \mathrm{a} : \alpha \Rightarrow \hat{\Gamma} \vdash (\mathrm{f} \triangleright \mathrm{a}) : \beta$ ['parametric'] or

[$\triangleright\vdash\oplus$] $\hat{\Gamma}_1 \vdash \mathrm{f} : \mathrm{C}\alpha\beta, \hat{\Gamma}_2 \vdash \mathrm{a} : \alpha \Rightarrow \hat{\Gamma}_1, \hat{\Gamma}_2 \vdash (\mathrm{f} \triangleright \mathrm{a}) : \beta$ ['cumulative'].

In general, however, a rule of inference can be arbitrarily complex, so that the identification

(*general*) *rule of inference* = *witness* **operator** goes beyond the conventional views on 'algebraic operators'.[17] In order to accomodate, formally, the terminology – and the notation – , one can use the idea of a *generic arity* (gen-arity, for short), viewed as a finite sequence of *non-negative integers*, to be associated to an arbirary operator, taken in the new sense. In this setting, the algebraic [null-ary] constants would get gen-arity *nil* [= the empty sequence], the usual n-ary algebraic operations would get gen-arity [0, ..., 0] (n times 0, $n > 0$), the n-adic abstractors would get gen-arity [n] ($n > 0$), so that the monadic λ-abstractor, as well as the usual quantifiers, for that matter, must have gen-arity [1], the dyadic abstractor *split* [$\int$], mentioned incidentally below, has gen-arity [2], and so on. In particular, the 'mixed' operators (gen-arity [k_1, ..., k_n], $k_i > 0$) can be handled as 'flat' (algebraic) n-ary operations acting on k_i-adic abstractors $(0 < i < n + 1)$.

In practice, however, we rarely, if ever, encounter complex rules corresponding to 'mixed' operators; we are normally confronted with 'flat' (ordinary algebraic) operators or with n-adic abstractors with n := 1,2 (operators of gen-arity [1] or [2]), at most, so that, in the end, the talk about gen-arities amounts to a piece of empty generality.[18]

§3 Essentially, the Łukasiewicz Warsaw Lectures of 1928–1929, Łukasiewicz (1929), contain a very detailed axiomatic presentation of

(1) [classical] propositional logic, based on the signature [N,C] (*classical negation* and *material implication*, in Łukasiewicz notation)[19], and

(2) a mild – yet very clean – version of the 'extended propositional [classical] logic', i.e., the [classical] propositional logic with propositional quantifiers, à la Peirce (1885), Russell (1906) and Tarski (doctoral diss., Warsaw 1923, under Leśniewski), or else Leśniewski's 'protothetic', for that matter.[20]

Now, except for a minor detail, the latter one is not more than the former, because we can define explicitly the [classical] propositional quantifiers in terms of [classical] connectives K [*and*], A [*or*] (in Łukasiewicz notation), and propositional constants **v** [*verum*] and **f** [*falsum*], anyway (just 'truth value quantifiers', as Nuel Benap Jr would have had them[21]).

The 'minor detail' refers, here, to the fact that the primitive [N,C]-signature is not *functionally complete*: we cannot obtain the propositional constants **v** and **f** from [N,C] alone. This does not affect our discussion of (2) below, as the signature [N,C,Π], with Π for the *universal propositional quantifier*, is functionally complete and even redundant, since one can define

f and N à la Peirce (1885) by $\mathbf{f} := \Pi$p.p and Nα := Cα**f**, resp.[22] In an axiomatic quantifier-free setting – as in Łukasiewicz (1929), Chapter II – the absence of the propositional constants might, however, affect the translation of the axioms in terms of rules of inference (and conversely). The point is that we need a primitive **f** [*falsum*] in order to express something as simple as the 'law of (non-) contradiction', for instance, in inferential (entailment-like) terms.[23] By adjoining a *falsum*-constant **f** to [N,C], the Łukasiewicz original axiom system is, however, *incomplete* as it stands. We cannot even prove, from the Łukasiewicz axioms, the 'thesis' $\vdash$ **v** [$\equiv$ N**f**], for instance.[24]

Recall that the Łukasiewicz (1929) quantifier-free axioms, with *modus ponens* and substitution as only 'rules of inference', are (in Łukasiewicz – 'Polish' – spelling):

> $\vdash$ **CB**[p,q,r] := 1 : CCpqCCqrCpr
>
> [transitivity of implication: 'suffixing'] - axiom in Peirce (1885)
>
> $\vdash$ **E**[p] := 2 : CCNppp
>
> [the *consequentia mirabilis* of Gerolamo Cardano (1570) or the *Law of Clavius*, viewed as a 'thesis']
>
> $\vdash$ **O**[p,q] := 3 : CpCNpq
>
> [*ex contradictione quodlibet*, 'explosion'].

To this team, we add, for reasons discussed above:

> $\vdash \mathbf{\Omega}$: **v**.

§3.1 *Proof-combinators*. Taking **CB**, **E** and **O** – with the appropriate propositional parameters (here: [p,q,r,] [p], and [p,q], resp.) –, as well as $\mathbf{\Omega}$, in guise of primitive 'witnesses' for the corresponding axioms, detachment / *modus ponens* can be viewed as a binary (algebraic) operation $\mathfrak{D}$, from 'detachment' (to be defined properly – i.e., equationally – later on) acting on witnesses, to the effect that:

> ($\triangleright$ = *modus ponens*) if f is a witness for C$\alpha\beta$ and a is a witness for α, then $\mathfrak{D}$fa is a witness for β.

We write, for convenience, f(a) = (f $\triangleright$ a) := $\mathfrak{D}$fa. This is to be understood modulo arbitrary uniform substitutions, with the proviso that one must take most general substitutions into account (here, substitutions are endomorphisms of the corresponding [free] algebra).[25]

For the record, the formal grammar (for formulas, resp. *witness terms* [*proof-terms* or *w-terms*, for short]) is:

propositional variables :: p, q, r, ...

formulas :: α, β := p | Nα | C$\alpha\beta$

w-variables :: x, y, z, ...

w-terms :: a,b,c,d,e,f := x | $\mathbf{\Omega}$ | **CB** | **E** | **O** | c$\triangleright$a.

Where α is a formula and a is a w-term, we write, as ever, $\vdash$ a : α, for the fact that a is a witness (actually, a w-term) for α.

So, we have, in particular, *derived* rules (here, *definable witness-operators*):

$\vdash$ g $\circ$ f := **CB**[p,q,r](f)(g) : Cpr, if $\vdash$ f : Cpq and $\vdash$ g : Cqr,

$\vdash$ **E**[p](f) : p, if $\vdash$ f : CNpp,

$\vdash$ **O**[p](a)(c) : q, if $\vdash$ a : p and $\vdash$ b : Np.

Examples, ignoring propositional parameters on witnesses, as well as explicit substitutions[26]:

$\vdash$ 4 := **CB** $\triangleright$ **CB** : CCCCqrCprsCCpqs

$\vdash$ 5 := 4 $\triangleright$ 4 = (**CB** $\triangleright$ **CB**) $\triangleright$ (**CB** $\triangleright$ **CB**) : CCpCqrCCsqCpCsr

$\vdash$ 6 := 4 $\triangleright$ 1 = (**CB** $\triangleright$ **CB**) $\triangleright$ **CB** : [exercise]

$\vdash$ 7 := 5 $\triangleright$ 6 : [exercise]

$\vdash$ 8 := 7 $\triangleright$ 1 : [exercise]

$\vdash$ 9 := 1 $\triangleright$ 3 = **CB** $\triangleright$ **O** : [exercise]

...

$\vdash$ **I** := 16 = 9 $\triangleright$ 2 = (**CB** $\triangleright$ **O**) $\triangleright$ **E** : Cpp - axiom in Peirce (1885)

and so on.

Further, Łukasiewicz meticulously obtained

$\vdash$ **K** $:= 18 :$ CqCqp [‘the law of simplification’] –

axiom in Frege BS (1856)

$\vdash$ **CI** $:= 20 :$ CpCCpqq

[‘assertion’ or internalised *modus ponens*]

$\vdash$ **C** $:= 21 :$ CCpCqrCCqCpr

[‘the law of commutation’] - axiom in Frege BS, as well as in Peirce (1885) [*Note* by Łukasiewicz (cca 1925): superfluous in Frege BS, it can be already obtained from **K** and **S**.]

$\vdash$ **B** $:= 22 =$ **C** $\triangleright$ **CB** : CCqrCCpqCpr

[transitivity of implication: ‘prefixing’]

$\vdash$ **P** $:= 24 :$ CCCpqpp

[‘the Law of Peirce’] - axiom in Peirce (1885)

$\vdash$ **W** $:= 30 :$ CCpCpqCpq

[‘Hilbert’ or ‘contraction’]

$\vdash$ **S** $:= 35 :$ CCpCqrCCpqCpr

[‘Frege’ or ‘selfdistribution on the major’] - axiom in Frege BS

$\vdash \Delta := 39 :$ CNNpp

[‘law of double negation’ (elim)] - axiom in Frege BS

$\vdash \nabla := 40 :$ CpNNp

[‘law of double negation’ (intro)] - axiom in Frege BS

...

[46–49: ‘the laws of transposition’ (or ‘contraposition’)]

$\vdash 46 :$ CCpqCNqNp

$\vdash 47 :$ CCpNqCqCNp - axiom in Frege BS

$\vdash 48 :$ CCNpqCNqp

$\vdash 49 :$ CCNpNqCqp

etc.

This amounts to a ‘typed’ (stratified, decorated) *combinatory logic notation*, where one manipulates formulas in guise of so-called *principal type schemes*.[27]

Now, as Tarski should have known (in 1921), in presence of *modus ponens*, the Deduction Theorem (DT) – or implication-introduction – can be obtained from **K** and **S** alone. This yields the λ-calculus counterpart of the same story.

§3.2 (DT) *and the λ-abstraction-algorithm.* I have argued at length elsewhere that Tarski must have been, likely, familiar with *some form* of (typed) λ-calculus – or (typed) combinatory logic or both – during the early 1920, knowledge that enabled him to prove some tricky axiomatizability results around 1925.[28]

Indeed, there is, essentially, a *single way* of proving (DT): the proof amounts to a simple *inductive argument*.

The reasoning can be repeated in any (propositional) logic – with substitution and *modus ponens*, as only primitive rules of inference – that contains the (witnessed) 'theses':

$(\mathbf{K}) \vdash \mathbf{K}[\mathrm{p.q}] : \mathrm{CpCqp}$, and

$(\mathbf{S}) \vdash \mathbf{S}[\mathrm{p,q,r}] : \mathrm{CCpCqrCCpqCpr}$.

Note first that, in such cases, one has, as derived rules:

$[\mathbf{K}]\ \hat{\Gamma} \vdash \mathrm{f} : \mathrm{p} \Rightarrow \hat{\Gamma} \vdash \mathbf{K}[\mathrm{p,q}](\mathrm{f}) : \mathrm{Cqp}$, and

$[\mathbf{S}]\ \hat{\Gamma} \vdash \mathrm{f} : \mathrm{CpCqr},\ \hat{\Gamma} \vdash \mathrm{g} : \mathrm{Cpq} \Rightarrow \hat{\Gamma} \vdash \mathrm{f} \,\Box\, \mathrm{g} : \mathrm{Cpr}$,

where $\mathrm{f} \,\Box\, \mathrm{g} := \mathbf{S}[\mathrm{p,q,r}](\mathrm{f})(\mathrm{g})$, as well as the (witnessed) 'thesis':

$(\mathbf{I}) \vdash \mathbf{I}[\mathrm{p}] : \mathrm{Cpp}$

[ignoring propositional parameters, the latter is available as $\mathbf{S}(\mathbf{K})(\mathbf{K})$].

Suppose that we have obtained a proof b[x] of β from the assumption that we have a proof x of α (so that b[x] depends possibly on [x:α]). Then (DT) states that we must have a proof $\lambda\mathrm{x}{:}\alpha.\mathrm{b}[\mathrm{x}] := \lambda([\mathrm{x}{:}\alpha](\mathrm{b}[\mathrm{x}]))$ of $\mathrm{C}\alpha\beta$, that does not depend on the proof [x:α], *ceteris paribus*.[29] That is to say, formally,

$(\lambda)\ \hat{\Gamma} \vdash \lambda\mathrm{x}{:}\alpha.\mathrm{b}[\mathrm{x}] : \mathrm{C}\alpha\beta$, if $\hat{\Gamma},\ [\mathrm{x}{:}\alpha] \vdash \mathrm{b}[\mathrm{x}] : \beta$,

for an appropriate assumption-context $\hat{\Gamma}$, as a parameter in the argument.

The induction pays attention to the form ('structure') of b[x]. To save repetitions, set $\mathrm{e} \equiv \lambda\mathrm{x}{:}\alpha.\mathrm{b}[\mathrm{x}]$. There are only three cases to examine:

(1) b[x] $\equiv$ [x : α]; so $\alpha \equiv \beta$; set e := $\mathbf{I}[\alpha]$: C$\alpha\alpha$;

(2) b[x] : β does not actually depend on [x : α]; set e := $\mathbf{K}[\beta,\alpha]$(b) : C$\alpha\beta$;

(3) b[x] $\equiv$ (f $\triangleright$ g) : β; where f : C$\alpha'\beta$ and a : α'; then the (IH) guarantees $\hat{\mathrm{f}}$:= (λx:α.f) : CαC$\alpha'\beta$ and â := (λx:α.a) : C$\alpha\alpha'$; set e := $\hat{\mathrm{f}}$ $\Box$ â : C$\alpha\beta$.[30]

§3.3 *An extended λ-calculus.* As in (decorated / 'typed') λ-calculus, we can thus write (ignoring everywhere proof-context parametrisations):

(λ) $\vdash$ λx:p.b[x] : Cpq, if [x:p] $\vdash$ b[x] : q.

As one might already guess, this makes up the first step in a would-be attempt meant to replace the Łukasiewicz axioms – i.e., the primitive combinator team $\{$**CB**, **E**, **O**$\}$ – together with $\mathbf{\Omega}$, on the signature [**f**, N,C], by appropriate witness operators (rules of inference).

Set now

(ϵ) $\vdash$ ϵx:Np.a[x] := **E**(λx:Np.a[x]) : p, if [x:Np] $\vdash$ a[x] : p.

The latter derived rule (definable witness-operator) is the *consequentia mirabilis* of Gerolamo Cardano (1501–1576) or the *Rule of Clavius*, viewed as a single-premiss rule of inference[31].

As announced before, we adjoin the propositional constant **f** (*falsum*), with **v** := N**f** (*verum*), and a single additional (witness) axiom:

($\mathbf{\Omega}$) $\vdash$ $\mathbf{\Omega}$: **v**

and set[32]

(ϖ) $\vdash$ ϖ[p](e) := **O**[**f**,p](e)($\mathbf{\Omega}$) : p, if $\vdash$ e : **f**,

with, finally

($\star$) $\vdash$ c $\star$ a := **O**[p,**f**](a)(c) : **f**, if $\vdash$ a : p, and $\vdash$ c : Np

['inner cut' or the 'rule / law of (non-) contradiction'].

Conversely, **E** and **O** can be obtained as

$(\mathbf{E}) \vdash \mathbf{E}[\mathrm{p}] := \lambda \mathrm{f{:}CNpp}.\epsilon \mathrm{x{:}Np}.(\mathrm{f}\star\mathrm{x}) : \mathrm{CCNppp},$

$(\mathbf{O}) \vdash \mathbf{O}[\mathrm{p,q}] := \lambda \mathrm{x{:}p}.\lambda \mathrm{y{:}Np}.\varpi[\mathrm{q}](\mathrm{y}\star\mathrm{x}) : \mathrm{CCpCNpq}.$

As is well-known, the rules (λ), $(\triangleright)$, (ϵ), (ϖ), and $(\star)$, with the additional axiom $(\mathbf{\Omega})$, suffice to yield full classical [propositional] logic.[33]

On the other hand, if $(\star)$ is present, the rules (ϵ) and (ϖ) of the [**f**,N,C]-signature, taken together, are equivalent, in this context, to *reductio ad absurdum*, (∂), viewed as a single-premiss rule

$(\partial) \vdash \partial \mathrm{x{:}Np.e[x]} : \mathrm{p}$, if $[\mathrm{x{:}Np}] \vdash \mathrm{e[x]} : \mathbf{f}$.

Indeed, one has

$(\partial) \vdash \partial \mathrm{x{:}Np.e[x]} := \epsilon \mathrm{x{:}Np}.\varpi[\mathrm{p}](\mathrm{e[x]}) : \mathrm{p}$, if $[\mathrm{x{:}Np}] \vdash \mathrm{e[x]} : \mathbf{f}$,

and, conversely,

$(\epsilon) \vdash \epsilon \mathrm{x{:}Np.a[x]} := \partial \mathrm{x{:}Np}.(\mathrm{x}\star\mathrm{a[x]}) : \mathrm{p}$, if $[\mathrm{x{:}Np}] \vdash \mathrm{a[x]} : \mathrm{p}$, and

$(\varpi) \vdash \varpi[\mathrm{p}](\mathrm{e}) := \partial \mathrm{x{:}Np.e}$, if $[\mathrm{x{:}Np}] \vdash \mathrm{e} : \mathbf{f}$ (x not free in e)[34],

so that, finally, classical [propositional] logic can be based on

(1) the axiom $(\mathbf{\Omega})$, and the four rules:

(2) (λ) [the 'Deduction Theorem', implication-introduction],

(3) $(\triangleright)$ [*modus ponens*, implication-elimination],

(4) (∂) [*reductio ad absurdum*], and

(5) $(\star)$ ['the law of (non-) contradiction'].

The axiom is, in fact, redundant, since, in this case, one can define explicitly:

$(\mathrm{df}\ \mathbf{\Omega}) \vdash \mathbf{\Omega} := \partial \mathrm{x{:}N}\mathbf{v}.\partial \mathrm{y{:}}\mathbf{v}.(\mathrm{x}\star\mathrm{y}) : \mathbf{v}.$

We shall keep, however, $\mathbf{\Omega}$ around for a while, mainly for the sake of comparison with the Jaśkowski (1934) version of 'natural deduction'.

The 'natural deduction' system above is easily seen to be equivalent to the axiomatics of Łukasiewicz (1929), modified as above such as to fit the primitive [**f**,N,C]-signature. As the rules have been already seen to be derivable from the axioms, this amounts to writing down the explicit definitions of the witnesses (here, *combinators*) **CB**, **E**, and **O** in terms of the proof-operators contained in the 'basis' $\{\lambda, \triangleright, \partial, \star\}$.

The corresponding (extended) λ-calculus is discussed next. It turns out that – if we forget about the constant $\mathbf{\Omega}$ – one can even formulate it in a decoration-free ('type-free') setting. This allows us establishing (its Post-) *consistency* in a straightforward way, using only some very basic λ-calculus facts.

§4 On the primitive [**f**,N,C]-signature, the minimal setting above – consisting of ($\mathbf{\Omega}$) [otherwise redundant], (λ) [implication-introduction], ($\triangleright$) [implication-elimination], (∂) [*reductio ad absurdum*], and ($\star$) ['the law of (non-) contradiction'] – can be viewed as an extension of the basic ('simple') typed λ-calculus $\boldsymbol{\lambda}$[C], obtained by 'replicating' its pure (λ)-($\triangleright$)-part.

Formally, the decoration-free ('type-free') syntax of the resulting $\lambda\partial$-calculus – $\boldsymbol{\lambda\partial}(\mathbf{\Omega})$, say – is given by:

witness-variables :: x, y, z, ...

witness terms :: a,b,c,d,e,f := x | λx.b | f$\triangleright$a | ∂x.e | c$\star$a.

In the resulting equational system, one has the usual $\beta\eta$-conditions for (λ) and ($\triangleright$) [decoration-free spelling]:

$(\beta\lambda) \vdash (\lambda \mathrm{x.b[x]}) \triangleright \mathrm{a} = \mathrm{b[x{:=}a]}$,

$(\eta\lambda) \vdash \lambda \mathrm{x.(c} \triangleright \mathrm{x)} = \mathrm{c}$ (x not free in c),

as well as the analogous $\beta\eta$-conditions for (∂) and ($\star$):

$(\beta\partial) \vdash \mathrm{c} \star (\partial \mathrm{z.e[z]}) = \mathrm{e[z{:=}c]}$,

$(\eta\partial) \vdash \partial \mathrm{z.(z{\star}a)} = \mathrm{a}$ (z not free in a), together with the expected rules of monotony (compatibility of equality – here, *conversion* – with the operations).

This extension of pure $\boldsymbol{\lambda}$ can be easily seen to be consistent by interpreting it in [type-free] $\lambda\pi$-calculus, $\boldsymbol{\lambda\pi}$, for instance.[35] Alternatively, one can choose to equip the resulting calculus with an appropriate *notion of reduction* and establish *confluence* [via a Church-Rosser theorem] first.

The *intended* decoration (typing) is given by the conditions (λ), ($\triangleright$), (∂) and ($\star$). In view of the above, if considered as a (decorated / 'typed') λ-theory, the outcome – the $\lambda\partial$-calculus $\boldsymbol{\lambda\partial}$[**f**,N,C] – is a *witness theory for classical logic*.

This yields the simplest *Curry-Howard correspondence for* (*propositional*) *classical logic* I know of.[36]

§5 In his (1927, 1934), Jaśkowski chose to hide the applications of the 'inner cut' ($\star$) – which, as noted above, would have required the additional propositional atom **f** (*falsum*) – and expressed *reductio ad absurdum* in the form of a more complex rule, viz. by the Medieval *ex contradictione quodlibet* ['explosion'] principle, viewed as a rule of inference:

$$(\chi)\ [\mathrm{z{:}Np}] \vdash \mathrm{c[z]} : \mathrm{Nq},\ [\mathrm{z{:}Np}] \vdash \mathrm{a[z]} : \mathrm{q} \Rightarrow\ \vdash \chi\mathrm{z{:}Np.(c,a)} : \mathrm{p}.$$

Upon adjoining the atom **f** and the 'hidden' rule ($\star$), the complex Jaśkowski rule (χ) can be obtained as:

$$(\chi) \vdash \chi\mathrm{z{:}Np.(c,a)} := \partial\mathrm{z{:}Np.(c{\star}a)} : \mathrm{p},$$
$$\text{if } [\mathrm{z{:}Np}] \vdash \mathrm{c} : \mathrm{Nq}, \text{ and } [\mathrm{z{:}Np}] \vdash \mathrm{a} : \mathrm{q},$$

while, conversely, one can have:

$$(\partial) \vdash \partial\mathrm{z{:}Np.e[z]} := \chi\mathrm{z.}(\boldsymbol{\Omega},\ \mathrm{e[z]}) : \mathrm{p}, \text{ if } [\mathrm{z{:}Np}] \vdash \mathrm{e[z]} : \mathbf{f},$$

in terms of (χ) and ($\boldsymbol{\Omega}$).

The 'hidden' rule ($\star$), however, can be obtained explicitly from Jaśkowski's (χ) only by an ad hoc contextual artifice, setting, e.g.,

$$(\star) \vdash \mathrm{c} \star \mathrm{a} := \chi\mathrm{z{:}}\mathbf{v}\mathrm{.(c,a)} : \mathbf{f}, \text{ if } \vdash \mathrm{c} : \mathrm{Np}, \text{ and } \vdash \mathrm{a} : \mathrm{p}.$$ [37]

As an aside, on ultimate formal grounds, I should have rather written down the Jaśkowski rule (χ), as:

$$[\mathrm{x{:}Np}] \vdash \mathrm{c[x]} : \mathrm{Nq};\ [\mathrm{y{:}Np}] \vdash \mathrm{a[y]} : \mathrm{q} \Rightarrow\ \vdash \chi(\mathrm{x,y{:}Np}).(\mathrm{c[x],a[y]}) : \mathrm{p},$$

but, as the two premisses of (χ) are independent, I am probably allowed to use (a subtle form of meta-) α-conversion in this context.[38]

From this, the reader can easily reconstruct by herself the witness theory corresponding to Jaśkowski's natural deduction system for classical logic, i.e., a would-be *Jaśkowski* $\lambda\chi\Omega$*-calculus* – $\boldsymbol{\lambda\chi\Omega}$[**f**,N,C], say – (equationally) equivalent to $\boldsymbol{\lambda\partial\Omega}$[**f**,N,C] above.

One might also note the fact that the original system of Jaśkowski (1927, 1934) – without **f** and ($\boldsymbol{\Omega}$), $\boldsymbol{\lambda\chi}$[N,C], say – was just a *notational device* (no proof-conversion, resp. proof-reduction rules). Moreover, it was constructed on a *functionally incomplete* propositional signature (as noted before, we cannot retrieve the constants **f**, **v**, definitionally, from N and C alone), whence the attempt to associate appropriate conversion-conditions to the Jaśkowski χ-primitive could only yield a *proper subsystem* of $\boldsymbol{\lambda\partial}(\boldsymbol{\Omega})$[**f**,N,C].

§5.1 Worth mentioning is also the fact that Jaśkowki proposed a perspicuous *graphical representation* of his proof-primitives in the original paper of 1927 – a kind of block-structure, meant to isolate intuitively *sub-proofs* of a given proof (actually, sub-terms in the corresponding λ-calculus description) –, that was perfected by Frederic Brenton Fitch (1908–1987) *et alii*, later on.[39]

Otherwise, the tedious and rather non-transparent formal description of the 'supposition rules' in Jaśkowski (1934) can be easily re-shaped, equivalently, in terms of assumption contexts and (witnessed) entailments as already suggested in the above. It is relatively easy to see that the usual 'structural' rules of Gentzen are implicit in Jaskowśki's description. Actually, Gentzen's L-system for classical logic is just a disguised form – namely, *a special case* – of 'natural deduction'.[40]

§5.2 Equally worth recording here is the (redundant) extension on the same primitive propositional signature [**f**,N,C], mentioned by the end of Rezuş (2009, rev. 2016), which consists of adding the double-negation (DN) rules:

($\boldsymbol{\nabla}$) $\vdash \boldsymbol{\nabla}$[p](a) : NNp, if $\vdash$ a : p [double-negation introduction],

($\boldsymbol{\Delta}$) $\vdash \boldsymbol{\Delta}$[p](c) : p, if $\vdash$ c : NNp [double-negation elimination].

In the latter case, the (DN) witness-operators [rules of inference] ($\boldsymbol{\nabla}$) and ($\boldsymbol{\Delta}$) are supposed to obey inversion principles of the form

$$(\beta\boldsymbol{\Delta}) \vdash \boldsymbol{\nabla}(\boldsymbol{\Delta}(\mathrm{c})) = \mathrm{c} : \mathrm{NNp},$$

$$(\eta\boldsymbol{\Delta}) \vdash \boldsymbol{\Delta}(\boldsymbol{\nabla}(\mathrm{a})) = \mathrm{a} : \mathrm{p}.$$

As earlier, the resulting extension (decorated / 'typed' λ-calculus, $\boldsymbol{\lambda\partial\Delta}$, say) can be shown to be consistent by interpreting its 'type-free' variant in the (undecorated) $\lambda\pi$-calculus.[41]

It is easy to see that, in the formulation without primitive (DN)-rules, at least one of the ($\beta/\eta\boldsymbol{\Delta}$)-conditions would normally fail, whence the idea of taking $\boldsymbol{\nabla}$ and $\boldsymbol{\Delta}$ as primitive proof-operators (rules of inference).

§6 The extensions to quantifiers (either propositional or first- resp. second order) are straightforward.[42]

Illustrated next is the extension to propositional quantifiers on the (otherwise redundant) signature [**f**,N,C,Π], with Π standing for the universal quantifier, like in Łukasiewicz (1929) and Jaśkowski (1934). As above, α, β, ..., possibly with sub- and / or superscripts are used as metavariables ranging over formulas. If the propositional variable p occurs free (even fictitiously so) in a formula α, we write α[p] in order to make this visible. Substitutions are mentioned accordingly: a[p:=α], and β[p:=α] resp. (read 'p becomes α in a, resp. in β').

For the extended witness-syntax there are required two more proof-operators (rules of inference), corresponding to *Generalization* (Λ) and *Instantiation* (►) resp. The new pair [(Λ),(►)] is analogous to the [(λ),(▷)]-pair above.

We present here a version close to Jaśkowski (1934), leaving to the reader the task of showing equivalence with the corresponding formulation of Łukasiewicz (1929). As above, the construction admits of a decoration-free description.

The decoration-free ['type-free'] syntax of the resulting system ($\boldsymbol{\lambda\partial\Lambda}$) is:

witness-variables :: x, y, z, ...

witness terms :: a,b,c,d,e,f := x | λx.b | f▷a | ∂x.e | c$\star$a | Λp.a | f►α.

The additional conversion rules are (in 'type-free' spelling):

$(\beta\Lambda) \vdash (\Lambda \mathrm{p.a[p]}) \blacktriangleright \alpha = \mathrm{a[p{:=}}\alpha]$,

$(\eta\Lambda) \vdash \Lambda \mathrm{p.(c} \blacktriangleright \mathrm{p}) = \mathrm{c}$, if p is not free in c[43],

together with the corresponding monotony conditions for (Λ) and $(\blacktriangleright)$, meant to make equality (conversion) compatible with the operations.[44]

The decoration ['typing'] is, as expected, relative to an arbitrary assumption context (i.e., a finite list $\hat{\Gamma}$ of decorated witness variables, omitted below). We have (λ), $(\triangleright)$, (∂), and $(\star)$, like before, as well as the new rules (for α, β arbitrary formulas):

$(\Lambda) \vdash (\Lambda \mathrm{p.a[p]}) : \Pi \mathrm{p}.\alpha[\mathrm{p}]$, if $[\mathrm{p}] \vdash \mathrm{a[p]} : \alpha[\mathrm{p}]$,

$(\blacktriangleright) \vdash (\mathrm{f} \blacktriangleright \alpha) : \beta[\mathrm{p{:=}}\alpha]$, if $\vdash \mathrm{f} : \Pi \mathrm{p}.\beta[\mathrm{p}]$.

The first- (resp. second-) order case is completely analogous. In each case, the corresponding λ-calculi can be shown to be consistent by simple translation arguments.

§7 The careful reader might have noticed a *general principle of construction* behind the *witness-theory* (proof-system) $\boldsymbol{\lambda\partial\Lambda}$ above, viz. the fact that the primitive witness- / proof-operators come in pairs [(**abs**),(**cut**)], where (**abs**) is a (monadic) *abstraction operator* and (**cut**) is a *'cut'-operator*, i.e., an operation meant to 'eliminate' its associated abstractor (**abs**). Moreover, each such a pair is supposed to *characterise* the associated rules of inference as *operators*, by *equational stipulations* (here, $\beta\eta$-conditions), i.e., more or less, algebraically, by indicating their 'characteristic behaviour'. One could thus notice a *uniform introduction-elimination pattern* (of construction), provided *one thinks in terms of witnesses* (here: proofs), *not in terms of* bare *formulas* (expressing propositions / propositional schemes).

Technically, it is also possible to describe a *proper extension* of the (minimal) *witness-theory* (proof-system) $\boldsymbol{\lambda\partial\Lambda}$, based on an idea that goes back to the founder of classical logic, the Stoic philosopher Chrysippus of Sol[o]i, twenty-two centuries ago. The extension is a λ-theory, i.e., a consistent λ-calculus, as well (both 'type-free', and decorated / 'typed' as above). Of course, I will not credit the famous Phoenician with the details, but the reader should be certainly able to recognise *the Chrysippean spirit* behind the construction.[45]

Writing down things in 'Polish' – i.e., in Łukasiewicz notation, as everywhere here –, I will use the same propositional signature as before, viz.

[$\mathbf{f}$,N,C,Π][46], but choose a slightly different team of primitive witness-operators, while leaving ∂, $\star$ and λ unchanged, add two kinds of 'pairs', namely π(...,...) and $\downarrow$(...,...), as well as a (mixed) dyadic abstraction-operator Σ, with term-forming rules $\prec$a,f$\succ$, $\downarrow_\alpha$(a) [writing, conveniently, $\prec$a,f$\succ$ $\equiv$ π(a,f) and $\downarrow_\alpha$(a) $\equiv$ $\downarrow$(α,a)], and Σ(p,x).c[p,x], resp., for proof- / witness-terms a, c[p,x], f and formulas α.

Whence the expected formal grammar [at a decoration-free / 'type-free' level], with p, q, r, ..., as (meta-variables for) propositional variables, as ever:

formulas :: α, β := p | Nα | C$\alpha\beta$ | Πp.α

w-variables :: x, y, z, ...

w-terms :: a,b,c,d,e,f := x | ∂x.e | c$\star$a | λx.b | $\prec$a,f$\succ$ | Σ(p,x).a | $\downarrow_\alpha$(a).

The (decoration-free / 'type-free') equational theory – called $\boldsymbol{\partial\lambda^*\Sigma}$, for convenience – consists of

($\beta\partial$) $\vdash$ c $\star$ (∂z.e[z]) = e[z:=c],

($\eta\partial$) $\vdash$ ∂z.(z$\star$a) = a (z not free in a),

as before, in $\boldsymbol{\lambda\partial}(\boldsymbol{\Lambda})$, and the following 'polar' conditions:

($\beta\lambda^*$) $\vdash$ $\prec$a,f$\succ$ $\star$ (λx.b[x]) = f$\star$(b[x:=a]),

($\eta\lambda^*$) $\vdash$ λx.∂y.($\prec$x,y$\succ$ $\star$ c) = c (x, y not free in c), as well as

($\beta\Sigma$) $\vdash$ $\downarrow_\alpha$(a) $\star$ (Σ(p,x).c[p,x]) = c[p:=α,x:=a]),

($\eta\Sigma$) $\vdash$ Σ(p,x).($\downarrow_p$(x) $\star$ c) = c (p, x not free in c),

together with the expected monotony constraints on the primitive witness-operators.

The *intended* decoration ('typing') is given by

(∂) $\vdash$ ∂x:Np.e[x] : p, if [x:Np] $\vdash$ e[x] : $\mathbf{f}$,

($\star$) $\vdash$ c$\star$a : $\mathbf{f}$, if $\vdash$ c : Nα and $\vdash$ a : α,

(λ) $\vdash$ λx:p.b[x] : Cpq, if [x:p] $\vdash$ b[x] : q,

like in the case of $\boldsymbol{\lambda\partial}(\boldsymbol{\Lambda})$, with moreover,

$(\pi) \vdash \prec$a,f$\succ$: NCpq, if $\vdash$ a : p, and $\vdash$ f : Nq, and

$(\Sigma) \vdash \Sigma$(p,x:Nα).c[p,x] : Πp.α, if [p] [x:Nα] $\vdash$ c[p,x] : **f**,

$(\downarrow) \vdash \downarrow_\alpha$(c) : N$\Pi$p.$\alpha$, if $\vdash$ c : Nα,

so that the classical 'polarities' become obvious.[47]

A few more (technical) comments are in order.

§7.1 It is easy to establish the fact that $\boldsymbol{\lambda\partial\Lambda}$ *is a subsystem of* $\boldsymbol{\partial\lambda^*\Sigma}$.[48] Indeed, define

(df $\triangleright$) c $\triangleright$ a := ∂y.($\prec$a,y$\succ\star$c), y not free in a and c.

This yields $(\beta\lambda)$ and $(\eta\lambda)$, i.e.

$(\beta\lambda) \vdash (\lambda$x.b[x]) $\triangleright$ a = b[x:=a],

$(\eta\lambda) \vdash \lambda$x.(c $\triangleright$ x) = c (x not free in c),

and, of course, monotony for the defined ($\triangleright$)-operator.

Set now

(df Λ) Λp.a[p] := Σ(p,z).(z$\star$a[p]), z not free in a[p],

(df $\blacktriangleright$) c$\blacktriangleright\alpha$:= ∂z.($\downarrow_\alpha$(z)$\star$c), z not free in c.

This yields $(\beta\Lambda)$ and $(\eta\Lambda)$, i.e.

$(\beta\Lambda) \vdash (\Lambda$p.a[p]) $\blacktriangleright$ α = a[p:=α],

$(\eta\Lambda) \vdash \Lambda$p.(c $\blacktriangleright$ p) = c, if p is not free in c.

as well as the expected monotony conditions for the defined [(Λ)-($\blacktriangleright$)]-pair of operators. The fact that the defined operators *inherit* the *intended* decoration ('typing') from the primitive decoration of the *definientia* is obvious.

§7.2 Incidentally, the (Σ-$\downarrow$)-free fragment of ['type-free'] $\boldsymbol{\partial\lambda^*\Sigma}$ – call it $\boldsymbol{\partial\lambda^*}$, for convenience – admits of an alternative, more general formulation [at a decoration-free level].

Indeed, setting

$$(\mathrm{df} \int)\ \int(\mathrm{x,y}).\mathrm{c}[\mathrm{x,y}] := \lambda \mathrm{x}.\partial \mathrm{y}.\mathrm{c}[\mathrm{e,y}] \text{ (for the } \textit{split} \text{ operator)},$$

we get, in $\boldsymbol{\partial\lambda^*}$,

$$(\beta \int) \vdash \prec\mathrm{a,b}\succ \star (\int(\mathrm{x,y}).\mathrm{c}[\mathrm{x,y}]) = \mathrm{f}\star(\mathrm{c}[\mathrm{x}{:=}\mathrm{a,y}{:=}\mathrm{b}]),$$
$$(\eta \int) \vdash \int(\mathrm{x,y}).(\prec\mathrm{x,y}\succ \star \mathrm{c}) = \mathrm{c} \text{ (x,y not free in c)},$$

together with the expected monotony conditions for $\int$ and it is obvious that we can trade $\int$ for λ in this context (at a decoration- / 'type-free' level), i.e., that one could have had, in the background, a calculus $\boldsymbol{\partial}\!\int$, say, instead of $\boldsymbol{\partial\lambda^*}$, in the above.[49]

The intended *art deco* would have been different, however. Actually, equational equivalence holds only in a decoration-free setting. For $\boldsymbol{\partial}\!\int$, one should change the primitive (propositional) signature, by replacing the primitive C [implication] with D [the 'Sheffer-functor' *incompatibility*, or *nand*, i.e. semantically, negated classical conjunction], whereupon N [classical negation] becomes redundant, by setting Np := Dpp. The resulting ['typed'] calculus $\boldsymbol{\partial}\!\int$[**f**,D], say – based on the witness primitives ∂, $\star$, $\int$ and the pair-construct π, as well as on the associated $\beta\eta$-conversion conditions ($\beta\partial$), ($\eta\partial$), and ($\beta \int$), ($\eta \int$), resp. – is, actually, a (proper) extension of $\boldsymbol{\partial\lambda^*}$[**f**,N,C], with Cpq := DpNq, in $\boldsymbol{\partial}\!\int$[**f**,D].[50]

§7.3 In the end, $\boldsymbol{\partial\lambda^*\Sigma}$ is (*Post*) *consistent*. For the (Σ-$\downarrow$)-free part of the proof, the result is already contained in **§7.2**. The genuine (Σ-$\downarrow$)-part consists of a trivial translation argument, collapsing the full system on its (Σ-$\downarrow$)-free fragment.

§7.4 One might also want to notice the fact that the analogous calculus $\boldsymbol{\partial}\!\int\!\boldsymbol{\Sigma}$[**f**,D,$\Pi$], based on {($\partial$), ($\star$), ($\int$), ($\pi$), ($\Sigma$), ($\downarrow$)} – as well as its corresponding (DN)-extension, for that matter – are 'polar' (Chrysippean) constructions, as well. Here, however, the details can be safely left to the reader.

§7.5 As a final (technical) remark, *all consistency proofs* mentioned in this paper amount to an easy – even though oft slightly involved – exercise of (explicit) *definability in (type-free)* $\lambda\pi$*-calculus* $\boldsymbol{\lambda\pi}$. Algebraically speaking, *we are dealing with* (a rather specific class of) *monoids* [viewed as algebraic varieties]. Since (the intuitionistically decorated) $\boldsymbol{\lambda\pi}$ is also known as 'the internal language of CCCs' [cartesian closed categories] among category theorists, most of the facts relevant here should also amount to *category theoretic folklore*.[51]

§8 *Coda.* I hope my discussion above has made more or less clear what Jan Łukasiewicz – and his (very) young student Stanisław Jaśkowski, as well as his (equally) young colleague Alfred Tajtelbaum [Tarski][52] – did *actually know* and / or *could have known*, as regards 'natural deduction', during the mid- and late twenties. Why they did not invent something like [decorated / 'typed'] λ-calculus, in order to make things *conceptually clean*, evades me completely. It's up to my better informed – and more gifted – readers to speculate upon.

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§9 Notes

[1]Stanisław Jaśkowski graduated from high-school at eighteen, in 1924, so he was about twenty, by then.

[2]The contrast 'Satz-' *vs* 'Regellogik' – roughly: 'sentence / proposition logic' [sic] *vs* 'rule logic' –, current in the German logico-philosophical literature, mainly after Gentzen, goes back to Frege (1893) and is meant to stress a difference of approach: *pace* Frege, the pioneers were mainly concerned with the formal study of *propositions* and /or *propositional schemes*, as expressed by *formulas*, and the properties thereof (like, e.g., *provability* in a given 'logistic' system ['Satzsystem'], etc.), while Frege and, subsequently, Gentzen payed also attention to the *rules of inference* and to their properties (like, e.g., *derivability* and / or *admissibility* in a given system [of rules]). With a suggestive term, we may refer to the former approach – and to its defendors / practitioners – as *Formularian* (with implicit allusion to Peano's various editions of his 'Formulaires', mere collections of [formalised] formulas). Roughly speaking, for a Formularian, a logic is a *set of provable formulas*, and a provable formula ['thesis' or, even, 'theorem'] is, at best, the codification of a [bunch of] rule[s] of inference. The Formularian approach has been effective in the early development of 'algebraic logic' and, later, in model theory, but is, conceptually speaking, rather inadvertent, since two distinct 'logics' may share exactly the same set of 'tautologies' [provable formulas], while still differring as to the corresponding derivable rules. The alternative *rule-oriented approach*, suggested by Łukasiewicz in his Warsaw Seminar (1926), was motivated in terms of 'naturality', by reference to the actual mathematical reasoning and this was, apparently, also the case for Gentzen, a bit later. Technically speaking, the distinction between *rule-admissibility* [closure of a set of propositions / formulas under a given rule of inference] and *explicit* [*rule*-] *derivability*, already implicit in Gentzen (1934–1935), comes rather late to the attention of the logical theorists; to my knowledge, it is due to Paul Lorenzen (1915–1994) – cf. Lorenzen (1955) – and to Haskell B. Curry (1900–1982), slightly later. The first *explicit* counterexample to the (Formularian) tenet that *a logic* = *a set of provable formulas*, is due to Henry Hiż (*circa* 1957–1958), who described a [three-valued] 'logic', $\mathbf{H_3}$ say, containing all classical, two-valued tautologies as provable formulas, where

not all classically valid rules of inference are derivable. See Hiż (1957, 1958, 1959) and, possibly, Nowak (1992), for a model-theoretical account of $\mathbf{H_3}$.

[3]For bio-bibliographical and historical details on Jaśkowski and his system(s) based on 'supposition rules' [Polish: 'oparta na dyrektywach załozeniowych'], i.e., proper / general rules of inference, using 'assumptions' in their premises, viz., entailments, see, e.g., Dubikajtis (1967, 1975) – a reliable informant on the logical whereabouts of Stanisław Jaśkowski, Lech Dubikajtis (1927–2014) was a former PhD student and a research assistant of Jaśkowski at the Warsaw National Institute for Mathematical Sciences (currently, the Institute of Mathematics of the Polish Academy of Sciences) –, as well as Kotas & Pieczkowski (1967), Orłowska (1975), Indrzejczak (1998, 2016), Piętka (2008), and, possibly, the textbook Borkowski & Słupecki (1963). On the subsequent history of 'natural deduction' – a comedy of conceptual errors, indeed –, see, e.g., Pelletier (1999, 2000), and Hazen & Pelletier (2012, 2014). General information on the 'Lvov-Warsaw school' can be found in Woleński (1985, 2015), Jadacki (2006), Wybraniec-Skardowska (2009), etc.

[4]Circa 1923. See the final outcome in Łukasiewicz (1934) and, possibly, Rezuş (2007, rev. 2016), for the main claim.

[5]On the authority of Paul Bernays (1888–1977), Gentzen borrowed his 'structural' rules from Hertz, a former physics student of Hilbert in Göttingen. Cf. Rezuş (2009, rev. 2016).

[6]See, e.g, the *Bibliography* and *Index of Names and Persons* of the collection Tarski (1956).

[7]In this respect, Frege was about two millennia behind Chrysippus (and, even, Aristotle, in a way). On this, see, e.g., Rezuş (2007, 2009, rev. 2016).

[8]As I could gather from the newest Russell expertise, this happened sometime around 1902–1903.

[9]As noticed, in passing, by Kurt Gödel, Russell was even later quite confused about the general concept of a *logical rule of inference*.

[10]Although he did not acknowledge the latter source. Cf. Russell (1906).

[11]One should perhaps read, once more, carefully, the rather vast output of Stanisław Leśniewski, on this. Cf. the collection Leśniewski 1992.

[12]See Axiom 8^* in Tarski (1930), and the footnote on page 32, in the collection Tarski (1956), for references. Some authors used to credit Herbrand with the discovery. However bright, Jacques Herbrand (1908–1931) was a teenager, just 13 years old, in 1921, so it is unlikely he spotted errors in Frege's [German] texts nobody used to read by then, even in Germany!

[13]At a quantifier-free level, Jaśkowski had a third rule – of the same kind

as (DT), actually –, yet a less inspired choice I will go into later on.

[14]In the limit case ($n = 0$), the axioms may be thought of as null-ary operations, if we want full generality.

[15]This was actually the case, historically speaking: the idea came first – exactly in these terms – to a later (Irish) student of Łukasiewicz, in the '(logical) Polish quarter' of Dublin, during the early fifties. [After the WWII, 'being unwilling to return to [...] Poland [...], Łukasiewicz looked for a post elsewhere. In February 1946 he received an offer to go to Ireland. On 4 March 1946 the Łukasiewiczes arrived in Dublin, where they were received by the Foreign Secretary and the Taoiseach Eamon de Valera. In autumn 1946 Łukasiewicz was appointed Professor of Mathematical Logic at the Royal Irish Academy (RIA), where he gave lectures at first once and then twice a week.' Simons (2014)] See the references to the $\mathfrak{D}$-operator, the 'condensed detachment' operator, of Carew A. Meredith, below, and, possibly the notes Meredith (1977) – by David Meredith, the American cousin of Carew, also a logician – for further historical details.

[16]Besides, one must have additional rules, called 'structural rules', in the current proof-theoretic terminology borrowed from Gentzen (1934–1935), that are rather trivial, and remain un-expressed, formally, in usual presentations of 'natural deduction' .

[17]The so-called 'abstraction-operators' – and, in general, the variable-binding mechanisms – are not welcome in (abstract) algebra, indeed. This on historical reasons, likely. Nicolaas G. de Bruijn observed once, in conversation, that abstraction-operators do not occur in pre-XIX-century mathematics. This explains, in a way, the initial lack of interest in such phenomena among algebraists. In recent times, when confronted incidentally with such cases – first-order quantifiers, for instance –, they made appeal to elaborated *local* solutions in order to cope with the problem. Paul Halmos and Alfred Tarski invented specific constructions – *polyadic algebras*, resp. *cylindric algebras* – in order to algebraise first-order logic with quantifiers, resp. quantifiers and equality. In more general situations, modelling abstraction operations in mathematical terms requires specific category theoretic methods and constructs that go far beyond the traditional algebraic way of thinking about 'operations' and 'operators'. Moreover, there are genuine phenomena, occurring frequently in computer science – like, e.g. the *non-local control* (typically, jumps), the *side effects*, or the so-called *continuations* – that can be easily described in terms of abstraction-operations, but whose behaviour resist algebraisation, as understood in traditional terms.

The (general) logical rules of inference fall within the same category.

[18]Like in the usual algebraic case, in fact, as we do not encounter 13-ary or 17-ary operations in current mathematical practice, either. — In logic, an exception can be encountered in the usual presentation of intuitionistic propositional logic, where the so-called *or*-elimination rule (case-analysis) is a witness operator of gen-arity [0,1,1], as well as in the case of Jaśkowski's rule χ (a witness-operator of gen-arity [1,1]) to be discussed below.

[19]Completeness is shown in Łukasiewicz (1929), Chapter III, **§22**. Cf. also Łukasiewicz (1931).

[20]Specifically, Chapter IV of Łukasiewicz (1929) is based 'in great part' [see the Preface of the first edition] on Tarski's previous work. Cf. also Łukasiewicz & Tarski (1930), **§5**. For Leśniewski, see now Leśniewski (1992).

[21]Cf., e.g., *mutatis mutandis*, Anderson & Belnap (1992) [**2**], **§33.4**.

[22]As actually done in Łukasiewicz (1929), Chapter IV, **§24**.

[23]The 'algebraic' alternative – which consists of [1] defining first a 'relative' *falsum* by $\mathbf{f}[\alpha] := \mathrm{NC}\alpha\alpha$, say, and [2] proving next $\mathrm{E}\mathbf{f}[\alpha_1]\mathbf{f}[\alpha_2]$, for any two fomulas α_1, α_2 etc. – induces unnecessary formal complications. A similar remark applies to Jaśkowski's 'natural deduction' quantifier-free system, based originally on [N,C] alone.

[24]It turns out that all we need is just a single new axiom $\vdash \mathbf{v}$, for this purpose. For completeness, see, for instance, Wajsberg (1937), I, **§5**, resp. 1939, II, **§2**, and the remark (below) that *ex falso quodlibet* can be obtained from the Łukasiewicz axiom $\vdash \mathbf{O}[\mathrm{p,q}] : \mathrm{CpCNpq}$ and a 'paradigmatic' proof of $\mathbf{v}$, like $\vdash \boldsymbol{\Omega} : \mathbf{v}$.

[25]Technically speaking, the $\mathfrak{D}$-operator is the 'condensed detachment' operator of Carew A. Meredith (1904–1976). Notably, the Irishman attended Łukasiewicz's lectures in Dublin, during the early 1950. See, e.g., David Meredith's bio-bibliographical note, Meredith (1977), and, possibly, Rezuş (1982, 2010), Kalman (1983), and Hindley & Meredith (1990), for details.

[26]The latter can be uniquely restored (modulo alphabetic variants) by the Robinson *unification algorithm*. Cf. Rezuş (1982).

[27]Cf. Hindley (1969, 1997), Hindley & Seldin (1986), and Hindley & Meredith (1990). As a matter of fact, here, one has a 'rigid' typing, à la Church and de Bruijn, instead. For the difference, see Hindley (1997), Barendregt *et al.* (2013) and the review Rezuş (2015). We could have had a (typed) *combinator theory* – a 'combinatory logic' –, as well, but, since the equational constraints on the primitive combinators are rather non-transparent, I prefer to skip the details. Otherwise, they can be recovered from remarks following

below.

[28]See, e.g, Rezuş (1982) and the discussion appearing by the end of Rezuş (2010).

[29]The *ceteris paribus* clause refers to the fact that the argument can be taken relative to a parameter $\hat{\Gamma} \equiv [\mathrm{x}_1{:}\alpha_1 \ldots \mathrm{x}_n{:}\alpha_n]$ $(n > 0)$.

[30]This is the so-called 'bracket abstraction algorithm' obtained first in terms of combinators – and, rather late, in this form –, by Haskell B. Curry (in 1948–1949) and, independently, by Paul C. Rosenbloom (1950, 2005^R), that is about thirty years after Tarski. See also Rosser (1942, 1953) and Curry & Feys (1958), **6**S.1, etc. One can improve on the last clause (3), by processing first the subcase $\mathrm{a} \equiv \mathrm{x} : \alpha \equiv \alpha'$, while setting $\mathrm{e} := \mathrm{f} : \mathrm{C}\alpha'\beta \equiv \mathrm{C}\alpha\beta$.

[31]Cf. Cardano (1570), Lib. **V**, Prop. 201, resp. Cardano (1663) **4**, p. 579. For *pater* Clavius [Christophorus Clavius, aka Christoph Klau, SJ (1537–1612)], cf. Clavius (1611) **1.1**, pp. 364–365 [comments *ad* Euclid *Elementa* **IX.12**], as well as **1.2**, p. 11 [comments *ad* Theodosius *Sphaerica* **I.12**]. See, also Rezuş (1991, rev. 1993), pp. 4, 23, 46, and Bellissima & Pagli (1996) *passim*, for details. Notably, Łukasiewicz was familiar with the references above, as well as with the medieval anticipations of his **O**-axiom (the 'Law of Duns Scotus'). Cf. e.g., Łukasiewicz (1929) and Łukasiewicz (1930), Chapter II, **§8**.

[32]This is the only place where we actually need $\mathbf{\Omega}$ in derivations.

[33]If the basis consists only of rules, as here, the axiom ($\mathbf{\Omega}$) is redundant. See below.

[34]No need for $\mathbf{\Omega}$, here. Cf., e.g., Rezuş (1990, 1991).

[35]This is possible since, unlike the pure λ-calculus $\boldsymbol{\lambda}$, the extensional $\lambda\pi$-calculus $\boldsymbol{\lambda\pi}$ contains infinitely many nontrivial copies of itself.

[36]See also Rezuş (1990, 1991, 1993) and Sørensen & Urzycyn (2006).

[37]In retrospect, it is hard to say why Jaśkowski did prefer the complex (χ)-rule (a kind of 'mixed' abstractor, in witness-theoretic terms, like the rather complex *case*-construct [*or*-elimination] in intuitionism), as a primitive rule of inference, in place of the 'elementary' *reductio ad absurdum* (∂) [here, a monadic abstractor, like (λ)] and the 'hidden' rule / operator expressing the 'law of [non-] contradiction' $(\star)$. *Prima facie*, I would suspect the choice was a matter of economy. Although there was an even more drastic economy in sight, that both Łukasiewicz and Jaśkowski were, apparently, well aware of, viz. by adopting the 'inferential' definition of negation, à la Peirce (1885), $\neg\mathrm{p} := \mathrm{Cp}\mathbf{f}$, in which case the primitive rule (χ) could have been replaced by

an 'inferential' variant of *reductio ad absurdum* (γ, a monadic abstractor, as well, with $\vdash$ γz:¬p.e[z] : p, for [z:¬p] $\vdash$ e[z] : **f**, in decorated / 'typed' version, etc.). Cf. Rezuş (1990, 1991, 1993). As a matter of fact, in the latter case, the witness-theoretic properties (as regards proof-conversion resp. proof-reduction [= detour elimination]) of the γ-operator are more involved that those presupposed by the 'natural' [(∂)-($\star$)]-pair, but Łukasiewicz and Jaśkowski did not think in such terms, anyway. Even Gentzen (1934–1935) was slightly confused as to the would-be proof-detours that could – and should – be associated to a genuine classical negation. It took us some thirty years, at least, until we were able to reach a clean conceptual insight on the matter. See, e.g., Prawitz (1965) for a solution, applying to the 'inferential' case and the combinator resp. λ-calculus variants, described in Rezuş (1990, 1991) [*$\lambda\gamma$-calculi*]. Besides, it took us about other twenty years, in order to get something as simple as the $\lambda\partial$-calculus sketched under **§4** above (Rezuş, cca 1987), corresponding to what the pioneers – Frege, Peirce, Russell, Łukasiewicz, Leśniewski, Tarski etc. – might actually have had in mind.

[38]Viewed abstractly, the witnessed entailments are, actually, a kind of *meta-combinators*, or *closed meta-terms*, in the end. — Incidentally, with the terminology mentioned earlier, the Jaśkowski witness-operator χ should have gen-arity [1,1], not gen-arity [2] (sic), whence the alternative spelling above.

[39]Cf. Fitch (1952) and Anderson & Belnap (1975, 1992), for applications to intensional logics. Notably, a similar representation was invented and used, later – independently –, by Hans Freudenthal (1905–1990), in didactic presentations of classical logic, as well as by Nicolaas G. de Bruijn (1918–1912), in his work on AUTOMATH [automated mathematics] and on the so-called 'Mathematical Vernacular' [WOT = *Wiskundige Omgangstaal*, in Dutch]. On this, see, mainly, de Bruijn's lectures on *Taal en structuur van de wiskunde* [The language and structure of mathematics], given at the Eindhoven Institute of Technology, Department of Mathematics and Computing Science, during the Spring Semester 1978, and summarised subsequently [in Dutch], in *Euclides 44* (1979–1980), as well as Rezuş (1983, 1990, 1991), for further references.

[40]A Gentzen L-sequent 'multiple on the right', α_1, ..., $\alpha_m \vdash \beta_1$, ..., β_n ($m, n \geq 0$), is a specific entailment of the form α_1, ..., α_m, $\bar{\beta}_1$, ..., $\bar{\beta}_n \vdash$ **f** – where $\bar{\beta}_i$ is a kind of 'surface negation' of β_i (for $0 < i < n+1$), a rather confusing idea based on an ad hoc piece of ideography –, also known as

rejection or *refutation* (*elenchos*, in the Greek of Aristotle and Chrysippus), about two millenia before both Jaśkowski and Gentzen were born. As a matter of fact, *mutatis mutandis*, Chrysippus' conceptual setting was cleaner. Cf. Rezuş (2007, rev. 2016) for details.

[41]So, once more, consistency can be established already at undecorated / 'type-free' level. In fact, $\boldsymbol{\lambda\partial\Delta}$ is redundant: if the $[(\boldsymbol{\Delta}),(\boldsymbol{\nabla})]$-pair is present, we can leave out either the $[(\lambda),(\triangleright)]$-pair or the $[(\partial),(\star)]$-pair, viz. $\boldsymbol{\lambda\partial\Delta}$ is, ultimately, equivalent [in a 'type-free' setting] with each of its 'halves', $\boldsymbol{\lambda\Delta}$, resp. $\boldsymbol{\partial\Delta}$.

[42]See, e.g., Rezuş (1990, 1991, 1993) for first-order quantifiers, and, *mutatis mutandis*, Rezuş (1986) for the 'extended propositional calculus' case (i.e., classical logic with propositional quantifiers), as well as for the second-order case.

[43]Exactly as in Girard's 'System F' (PhD Diss., Paris 7, 1971). Of course, the latter λ-calculus is the-$[(\partial),(\star)]$-free fragment of $\boldsymbol{\lambda\partial\Lambda}$, i.e., $\boldsymbol{\lambda\Lambda}$, by present notational standards. Cf. Rezuş (1986) for details on the Girard-Reynolds λ-calculus.

[44]Since I have omitted, everywhere in the above, any reference to proof-contexts, the usual provisoes on p-variables are also tacitly assumed.

[45]Cf. Rezuş (2007, rev 2016), for technical – and historical – evidence supporting the claim. The extension works for the system with (DN)-primitives, too.

[46]We may want to abbreviate, for convenience, $\bar{\mathrm{C}} := \mathrm{NC}$ and $\bar{\Pi} := \mathrm{N}\Pi$ (so that $\bar{\mathrm{C}}$ is marked as the 'polar [opposite]' of C, and $\bar{\Pi}$ as the 'polar [opposite]' of Π, resp.), but the Łukasiewicz notation makes this superfluous. One might also note the fact that, by the standards of Rezuş (2007, rev. 2016), $\bar{\mathrm{C}}$ and $\bar{\Pi}$ would have counted as *Chrysippean connectives*. Specifically, $\bar{\mathrm{C}}$ corresponds to the Chrysippean connector (binary connective) *more*, i.e., *māllon... ē...*, a kind of *rather... than...*, in English, while the quantifier-free part of the extended calculus [with (DN)-primitives] – to be described next – corresponds exactly to the *semantic (C-$\bar{\mathrm{C}}$)-fibration* of 'Chrysippean logic' **Ch**.

[47]Formally, ∂ looks, in the end, like a kind of degenerated Σ (sic). The informed reader has already realised the fact that the $(\Sigma\text{-}\downarrow)$-rules are just (undecorated / 'type-free') analogues of the usual *intuitionistic* $\exists$-*rules*. Cf. Rezuş (1986, 1991), etc. — On the historical side, if I am not very mistaken, I remember having encountered something similar to the 'polar' pair $[(\lambda)$-$(\pi)]$ in work of Dag Prawitz, going back to the late nineteen-sixties and the early seventies (although with no reference to the Stoic lore and / or

to would-be [classical] proof-isomorphisms, i.e., to proper proof-conversion rules). Whence, ultimately, the basic idea behind the construction of $\boldsymbol{\partial\lambda}^*(\boldsymbol{\Sigma})$ should be, very likely, accounted for as a piece of *(historical) data-retrieval*, rather than as a genuine finding, due to the present author. In retrospect, virtually *any mindful reader* of Prawitz, already familiar with the basics of λ-calculus, *could have came out with a similar proof-formalism*, even ignoring the Chrysippean antecedents.

[48]To show that $\boldsymbol{\lambda\partial\Lambda}$ is a *proper* subsystem of $\boldsymbol{\partial\lambda}^*\boldsymbol{\Sigma}$ requires a more involved argument. I'd rather defer the details (too far from the subject of the present notes, anyway).

[49]As a bonus, for $\boldsymbol{\partial}\int$ (Post-) consistency is straightforward. The latter is a (proper) subsystem of $\boldsymbol{\lambda\pi}$: define $\int$ by $\int$[x,y].c[x,y] := ∂z.c[x:=**1**(z),y:=**2**(z)], where **j**(c), j := 1, 2, are the usual $\boldsymbol{\lambda\pi}$-projections and $\partial \equiv \lambda$. — The $\lambda\pi$-calculus is known to be consistent by a well-known lattice-theoretical (actually topological) construction due to Dana Scott (1969), as well as by constructive ('syntactical') means, as shown recently by Kristian Støvring (November 2005, rev. 2006).

[50]In his Warsaw lectures, Łukasiewicz alluded actually to the alternative – cf., e.g., Łukasiewicz (1929), Chapter II **§17** –, but he was, apparently, distracted by *provability* details on Henry M. Sheffer (1913) and Jean Nicod (1917), so that the idea was diluted, later on. It is only in (very) recent times that the Peirce-Sheffer *nand* and *nor* connectives deserved a proper treatment in 'natural deduction' terms.

[51]The specific subject – falling under the label *cartesian closed monoids* [CCMs, for short] – has been invented by Dana Scott and Joachim Lambek sometime during the 1970's and has been vastly explored since, mainly in research on *categorical models of λ-calculus*.

[52]See Rezuş (1982, 1990) for relevant complements of information.

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