BEYOND BHK

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TEXT

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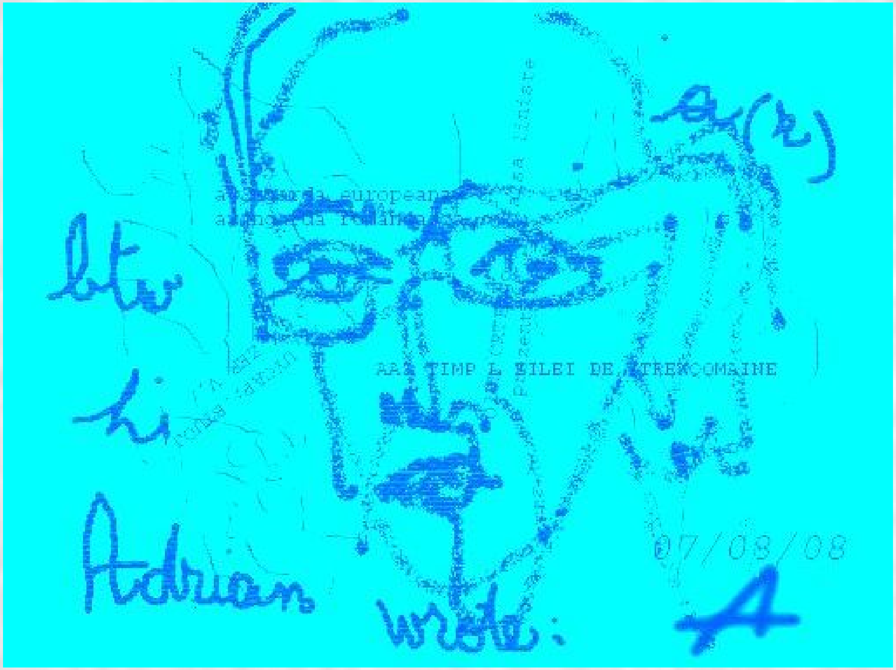
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Adrian wrote:
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Beyond BHK[0]

0. *Introduction. On a misleading tradition.* Traditionally, the "language" of the first-order classical logic has means of referring to *"individuals"* in some domain (so-called *univers du discours*) and means of expressing "facts" (or *"propositions"*) about them. The *proofs themselves* are handled in a *non-objectual* way. In particular, they are not codified syntactically. Some other kind of representation is implicitly involved in the so-called "proof theory". Usually, the latter makes appeal to the *eye* (as opposed to the *mind*!), rendering any *theoretical approach to logic* debatable.[1] This policy, justified, now and then, by an essentially empiricist *metaphysics of vision* [paradigmatically: Wittgenstein, but also the early Kreisel] is common in the post-Fregean logic tradition. Among other things, it forbids subjects like "proof-semantics for classical logic", for instance. On the same basis, it does not make too much sense to

[0] "Extended abstract". A reprint from: Henk Barendregt, Marc Bezem & Jan Willem Klop (eds.) **Dirk van Dalen Festschrift**, Utrecht 1993, pp. 114–120 [*Quaestiones Infinitae V – Publications of the Department of Philosophy, Utrecht University*] The full paper ["monograph"] is [likely] published elsewhere. Besides the obvious historical debts in matters of concern below, the author is, directly or indirectly, indebted to several persons, among whom (in, more or less, a *subjective* historical order): Corrado Böhm, Ernst Engeler, Dag Prawitz, Nuel D. Belnap Jr., Roger J. Hindley, Jon [= Jonathan P.] Seldin, Anne S. Troelstra, Henk [= Hendrik Pieter] Barendregt, Dirk van Dalen, David Meredith, Alonzo Church, Jeff [= Jeffrey] Zucker, Dick [= Nicolaas G.] de Bruijn, Bert [= L. S.] van Benthem Jutting, Diederik [= D. T.] van Daalen, Rob [= R. L.] Nederpelt, Peter [= P. A G.] Aczel, Per Martin-Löf, Martin [= Marteen] Bunder, Bob [= Robert K.] Meyer, Matthias Felleisen, and – [very] last[ly], but not least – Jean-Yves Girard.

[1] As understood here, logic is a *theory about proofs*, codifying, basically, our intuitions on *inference* and *generality*.

look for *theoretical criteria of proof-identity*, beyond obvious isomorphisms that can be established on "proof-figures", "proof-trees" or some other *ad hoc* organization of the visual representation space.

The [Frege]-Hilbert-Gentzen proof-theoretic tradition is, in fact, almost exclusively concerned with *provability*.[2] As useful as it might have been on a pure epistemological level (as opposing various metaphysical, psychological, or linguistic views on the "nature of proving", for instance), the "positive" investigation of the combinatorial properties of formula-configurations and the resulting discipline (the "metamathematics", as understood in the Hilbert school) remained parochial enterprises: *proof-structures* and *proof-properties* were thereby ignored programatically.[3]

There is room for a *mathematically specific* way of thinking about proofs, however, going beyond the elementary tree-mathematics. We are – surprisingly, indeed – able

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[2]This is a property of *formulas* (or, if one prefers, one of *propositions*), not a *proof-property*. Needless to say that model-theorists have been significantly more successful, so far, in extracting mathematically specific information on this matter.

[3]For the historian, the picture is somewhat more complex, since, as early as in 1904 [at the Third Int. Math. Congress], building upon his previous experience in axiomatics [for geometry], Hilbert can be seen to adopt, methodologically, an *amalgamating standpoint* in metamathematical research, advocating explicitly the need for a *"partially simultaneous development of the laws of logic and arithmetic"* [read: *"analysis"*]. As a side-effect, the new discipline comes to be confused, from the very beginning, about [the nature of] *logic* and/or its *specifics*, as contrasted with bare "mathematical" topics. (Although the factual pre-history of the so-called "Hilbert program" has been studied recently in detail [cf. Volker Peckhaus: Ph D Diss., Erlangen-Nürenberg 1990], the wide implications of this *"completely new aspect"* of Hilbert's thought – as Paul Bernays used still to think of it, in the early thirties – for the contemporary logic and proof-theory deserved little attention so far.)

to *recognize* and *identify* proofs on *pure theoretical* grounds. But the objects we are looking for, their structure and formal behavior, are to be properly located, at a higher "level of abstraction", so to speak.

The notes summarized here are meant to suggest a possible way of revising the traditional [= post-{Frege, Hilbert, Gentzen}] picture of the facts.[4]

1. *Categories of objects. Logics as equational proof-theories.* Beside individual terms and formulas, we construct first-order formal systems ("languages") with an additional "syntactic" category: the *proof-terms*. The intended meaning of a proof-formalism is given by the *propositions-as-types* isomorphism [H. B. Curry, C. A. Meredith,

[4]The somewhat older idea of a "general proof-theory" [D. Prawitz] won't perfectly cover the outcome, because the enterprise is – *prima facie*, at least – not a *philosophical* one. Actually, in recent times, some philosophers used to discuss closely related matters under the somewhat misleading rubric "meaning theory (for the logical operations)", thought of as a part of the *philosophy of language*. Still, the "linguistic" import of these topics is rather minimal, and Brouwer was right, in a sense, to fight against and discard this old – bi-millenial, in fact – illusion. Since, in view of the Hilbert tradition, the term "proof theory" ["reductive" sense] looks rather compromised by now (*pace* Kreisel), this area of investigation – making up, in a sense, "the core of logic", *qua* mathematical theory of proofs – appears to be in need for a right label. Our obvious *ideological* debts go back to L. E. J. Brouwer, A. Heyting and A. N. Kolmogorov (whence "**BHK**"). *Technically*, we improve on the pioneering work of Dag Prawitz [Ph D Diss., University of Stockholm 1965], exploiting also – among other things – deep intuitions of V. I. Glivenko (1928-1929) on the *nature of proving* in classical logic [*sic*] (whence, somehow, "beyond"). The "elementary" part of the discussion – as seen from the point of view of the worker in (typed) λ-calculus – appears in a forthcoming monograph of the author, on *the equational theory of classical proofs*. A larger body of facts (to be catalogued under the "right label" we are still looking for) will be surveyed next.

H. Läuchli, D. S. Scott, W. A. Howard, etc.]: the propositions are the "types" of proofs (*proof-classifiers*).

A *category of objects* is (analogous to) a constructive set (*à la* Errett W. Bishop or Per Martin-Löf). In intuitionistic/constructive mathematics, the proofs (traditionally: "derivations") make up a category of *abstract* objects. In this sense, understanding proofs would require

1° (generic) means to refer to *arbitrary objects* of the category (*proof-variables*, *proof-terms*) and

2° *identity-criteria* for these objects.

The proofs of a first-order logic L (containing the so-called "positive" implication) can be described *via* an equational theory $\lambda(L)$, viz. a typed [= "stratified"] λ-calculus, extending the ordinary typed λ-calculus λ^{τ}. So far, this procedure has been known to apply only to the first-order intuitionistic logic **HQ** and some derivatives, as obtained by adding, e.g., [propositional] quantifiers to the propositional fragment of **HQ** or higher-order classifiers ("universes"). As applied to **HQ**, this is the essence of the so-called "**BHK**"-interpretation ["**B**rouwer-**H**eyting-**K**olmogorov"] of the intuitionistic proof-operations [cf., e.g., A. S. Troelstra & D. van Dalen **Constructivism in Mathematics**, Amsterdam 1988]. We can extend this technique "beyond **BHK**", to classical first-order logic **CQ**, modal logics contained in Lewis' [first-order] **S5**, various intensional logics sharing genuinely classical [= "non-intuitionistic"] features, etc. [see footnote 4, above].

2. *Which proofs are "reliable" within the classical proof-world?* Here, we examine the equational proof-theory of the first-order intuitionistic logic ["the Heyting calculus"] *as a subsystem of classical proofs*.

In other words, we ask rather, in "positive" terms: *what is a "reliable" proof [for Brouwer and Heyting*, Dutch: *betrouwbaar*], *within the classical proof-world?* Obviously, the extant (classical, non-intuitionistic, post-Gentzen) tradition in proof-theory (*Beweistheorie*) has no means to answer this type of question. (In fact, the question cannot be even formulated properly.)

For an intuitionist, a *proof* is an (abstract) *object* (of thought), occurring naturally in the current (mathematical) practice, as a result of a *systematic reflection* on this practice. As expected, intuitionistically, a *proof-theory* is a piece of (intuitionistic) mathematics.

Putting aside the reputedly obscure ideological talk on "reliable" [read: "intuitionistic"] proofs *as "constructions"* in Brouwer, this view can be, *mutatis mutandis*, accommodated to the full proof-realm [of classical logic]: we can take it, indeed, as a *methodological guide* in the reconstruction of [classical] logic [= proof-theory, in some sense] as such.

3. *[Classical]* ***reductio ad absurdum*** *as an abstraction operator*. For the first-order classical logic **CQ** or its extensions, the required additions – to some background knowledge of typed λ-calculus – presuppose the fact that [1°] we are able to define, e.g., a proof-operator [an abstractor, say, i.e., a *proof*-variable binding mechanism], recording the genuinely Boolean uses of *reductio ad absurdum* and that [2°] we can describe its equational behavior. (There is a *combinatory* alternative to this, less

transparent, however.)

This operator points out to a new abstraction operation (here: the *γ-abstraction*): intuitively, it allows to conclude "positively" to the fact that a proposition expressed by a formula A has a proof [γx:¬A.e⟦x⟧, say] from the fact that a proof [e⟦x⟧] of a contradiction ⊥ [= *falsum*, *absurdum*] has been obtained from the ["negative"] assumption that there is an arbitrary proof [x] of ¬A. (Everywhere here, negation ¬ is supposed to be understood inferentially [$\neg A \equiv_{df} A \to \bot$].)

The general concept disclosed is that of a [*typed*] *λγ-theory* [= Post-consistent extension of λ^{τ}]. In particular, the proof-theory of the first-order classical logic **CQ** can be formulated as a typed λ-calculus λ(**CQ**).

We build on a typed λ-calculus $\lambda\pi!$, which is familiar from the *Automath* literature and Martin-Löf's type-theory: beside the usual typed abstractions and applications, $\lambda\pi!$ has *type-products* [represented by conjunctions] with an *extensional pairing* ["surjective pairing"] and is augmented by first-order abstractors and applications associated to *first-order products of families of types* [represented by universal quantifications]. In fact, $\lambda\pi!$ is, more or less, the "pure" part of an *Automath*-system, proposed by Jeffrey Zucker in 1975. Proof-theoretically, it describes also the proof-behavior of the [⊥,→,∧,∀]-fragment of Johansson's *Minimalkalkül*. The (typed) calculus $\lambda\pi!$ is known to be Post-consistent [*qua* equational theory].

4. $\lambda\gamma_{(0,\&)}$CQ: *the ["actualistic"] proof-theory of first-order classical logic***.** The most economical formulation for a λ(**CQ**)-theory is likely a calculus $\lambda\gamma_0$**CQ**, which extends properly $\lambda\pi!$, by primitive γ-abstractions γx:¬A.e⟦x⟧, where A is *atomic* ("prime"), subjected to the obvious proof-term *stratification rule*:

$$(\rightarrow i\gamma)\text{: } \Gamma[\mathrm{x}{:}\neg \mathrm{A}] \vdash \mathrm{e}[\![\mathrm{x}]\!] : \bot \Rightarrow \Gamma \vdash \gamma \mathrm{x}{:}\neg \mathrm{A}.\mathrm{e}[\![\mathrm{x}]\!] : \mathrm{A},$$

for any assumption-set Γ, and – ignoring γ-congruence – only two equational postulates, stating, resp.

$[\eta \rightarrow \gamma]$: γ-*extensionality*, i.e., unicity of the γ-behavior relative to the usual typed applications [= uses of *modus ponens*], [formally, assumming stratifiability on both sides, one has a reversal of the usual η-rule:

$$\gamma \mathrm{x}{:}\neg \mathrm{A}.\mathrm{x}(\mathrm{f}) = \mathrm{f}, \text{ provided f does not depend on x], and}$$

$[\oint \gamma]$: γ-*diagonalization*, allowing to eliminate specific uses of *reductio ad absurdum* occurring within the scope of ["inside"] a proof by *reductio* "of the same type", so to speak. [Assumming stratifiability on both sides, this reads, formally:

$$\gamma \mathrm{x}{:}\neg \mathrm{A}.\mathrm{f}[\![\mathrm{x}]\!](\mathrm{x}(\gamma \mathrm{y}{:}\neg \mathrm{A}.\mathrm{e}[\![\mathrm{x},\mathrm{y}]\!])) = \gamma \mathrm{z}{:}\neg \mathrm{A}.\mathrm{f}[\![\mathrm{z}]\!](\mathrm{e}[\![\mathrm{z},\mathrm{z}]\!]),$$

where the proof-term $\mathrm{e}[\![\mathrm{z},\mathrm{z}]\!]$ is obtained from $\mathrm{e}[\![\mathrm{x},\mathrm{y}]\!]$ by identifying the displayed proof-variables $[\mathrm{x} \equiv \mathrm{y} \equiv \mathrm{z}]$. Intuitively, a proof represented by a proof-term of the form

$$\gamma \mathrm{x}{:}\neg \mathrm{A}.\mathrm{f}[\![\mathrm{x}]\!](\mathrm{x}(\gamma \mathrm{y}{:}\neg \mathrm{A}.\mathrm{e}[\![\mathrm{x},\mathrm{y}]\!]))$$

has, indeed, the character of a slightly sophisticated "proof-détour", since γ plays, classicaly, the rôle of an "introduction" rule, rather than that of an "elimination". However, the "*int*roduction-*elim*ination" dichotomy applies properly only to *Minimalkalkül*-like systems of rules/proof-operators.]

Note that A is supposed to be "prime", in both $[\eta \rightarrow \gamma]$ and $[\oint \gamma]$.[5]

[5] For readers familiar with the proof-system of the Heyting first-order logic, the γ-diagonalization postulate must be also reminiscent of the Heyting "commuting conversions". Indeed, the "selectors"

The "complex" uses of *reductio* [γ-abstractions $\gamma x{:}\neg A.e[\![x]\!]$, where $A \equiv \bot$ or a complex formula] can be then introduced by an *inductive definition*.

The full theory $\lambda\gamma_{\&}$**CQ**, obtained by using arbitrary typed γ-terms as primitives, and by replacing the inductive conditions ([$\gamma\bot$], [$\gamma\!\rightarrow$], [$\gamma\wedge$], [$\gamma\forall$]) by obvious postulates is (stratification resp. equationally) equivalent to $\lambda\gamma_0$**CQ**.[6]

We can show **Cons**($\lambda\gamma_0$**CQ**), i.e., Post-consistency for $\lambda\gamma_0$**CQ** [= proof consistency for first-order classical logic], by extending the kernel of the "negative" translation of V. I. Glivenko [**Acad. Royale de Belgique, Bull. Cl. Sci.** (5) **15**, 1928, pp. 225–228] to the proof(-term)s of $\lambda\gamma_0$**CQ**. The outcome is an *effective* (1–1) *translation* from $\lambda\gamma_0$**CQ** to its γ-free fragment $\lambda\pi!$, known to be Post-consistent.[7] In particular, the procedure – described on half a page – is also *admissible* intuitionistically: it supplies a "dictionary" for proof-operations that do not make sense, in general, in the intuitionist practice.

What is *not intelligible intuitionistically* is just the *intuitive interpretation of the abstract γ-operations* as *logical proof-operations*: indeed, the abstraction operator corresponding, classically, to *reductio ad absurdum* has only *local* – "finitary", so to speak – *meaning* in terms of **HQ**-proofs. Technically, for each [$\bot,\rightarrow,\wedge,\vee,\forall,\exists$]-formula A and any proof f such that f proves intuitionistically $\neg A \vee A$, there is an abstraction

associated to the intuitionistic $\vee$ and $\exists$ are analogous to appropriate uses of γ. In general, the Heyting proof-calculus would allow only "cancelling" γ-abstractions, of the form $\omega_A(e) \equiv_{df} \gamma x{:}\neg A.e$, where $e : \bot$ does *not* "depend on" the assumption [x:¬A], so that this structural analogy is useful only in a genuinely Boolean setting.

[6]The "recursive eliminability" of γ was, in fact, known to Dag Prawitz (1965), although not in type-theoretic – or $\lambda\gamma$-calculus – form. This does not depend on γ-diagonalization, by the way.

[7]The result can be also obtained by a *type-free* argument.

operation γ_f, depending on f, which can be shown to share relevant properties with the "global" γ-abstractor. Beyond the – intuitionistically – recognizable "local" / "finitary" information, the classical γ-abstractor attempts to supply "global" information about proofs, acting, *qua* information-processing agent, like a *highly predictable* – although *intuitionistically "unreliable"* – *oracle*.

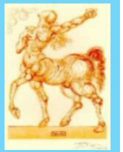
The calculus $\lambda\gamma_{\&}$**CQ** has [∨,∃]-"type-constructors" defined *via* the usual Ockham/De Morgan transformations [$A \vee B \equiv_{df} \neg(\neg A \wedge \neg B)$, with ∃ "generalizing" ∨, as expected]. These definitions admit of associated "negative" proof-operators (Boolean "injections" and Boolean "selectors"), with an appropriate *extensional* behavior. The latter appear to be more general than the Johansson-Heyting "injections" and "selectors", familiar from the literature on constructive type theories.

5. *The [full] Heyting calculus is a proper fragment of* $\lambda\gamma_{\&}$**CQ.** The bulk of the work is devoted to the derivation of the stratification and equational behavior of the *Minimalkalkül* resp. Heyting proof-operators associated to [⊥,∨,∃] in terms of the "negative" Boolean analogues. The *full* Heyting first-order proof-calculus λ**HQ**, with "⊥-conversions" [*ex-falso*-rules, here ω-*rules*] and so-called "commuting conversions" is shown, finally, to admit of a definitional embedding into the classical theory $\lambda\gamma_{\&}$**CQ**.

The simulation of the intuitionistic proof-operations in terms of $\lambda\gamma_{\&}$**CQ** proof-operations is an *effective translation*. This yields **Cons**(λ**HQ**), too.

Ultimately, we obtain an effective translation of λ**HQ** into $\lambda\pi!$, i.e., into a *proper fragment* of it, known to be consistent by intuitionistically acceptable means.

In fact, $\lambda\pi!$ and λ^{τ} can be shown to be *equi-consistent* by simple translations, so that **Cons**(λ**HQ**) amounts to **Cons**(λ^{τ}), i.e., the problem is reduced to a question



about the ordinary typed λ-calculus.[8] Existing alternatives of showing **Cons**(λ**HQ**) recommend either a *confluence* argument or a *model-theoretic* approach. The former involves hundreds of separate cases to check, whereas the latter has meager chances of becoming intuitionistically intelligible.

On the intuitive level, $\lambda\gamma_{\&}$**CQ** appears, in fact, as a kind of "asymptotic" extension of the first-order Heyting proof-calculus λ**HQ**, and corresponds – more or less – to an *"actualistic" interpretation of the classical logic proof-operations* [where "actualistic" = Gentzen's *"an sich"* in, e.g., **Math. Annalen 112**, 1936].

6. *The logic of "complete refutability"*. In analogy with λ**HQ**, one can isolate – within $\lambda\gamma_{\&}$**CQ** – the equational proof-theory of the first-order logic of "complete refutability" **DQ**, also known as a "logic of strict negation" [= Curry's logic *L***D**, in **JSL 17**, 1952, pp. 35–42]. The interest in **DQ** is in the fact that its inferential [$\perp$,$\to$] part is, in a sense, "non-Brouwerian" and – inside **CQ** – *complementary* to the inferential part of **HQ**. Indeed, **DQ** and **HQ** disagree mainly on negation: **DQ** allows "the Law of Clavius" [*consequentia mirabilis*: $\neg A \to A \to A$], which is, clearly, not **HQ**-derivable,

[8] Note that **Cons**(λ**HQ**) is *not* implied by the corresponding result for Martin-Löf's (1984) constructive type theory. Actually, the Heyting calculus is *not* contained *equationally* in any one of Martin-Löf's type theories, because of the "$\perp$-conversions" and the "commuting conversions", which are absent from the latter systems, even if higher order classifiers [= "universes"] and corresponding closure conditions are assumed. Equationally, indeed, Martin-Löf's type systems are just higher-order variations on the *Minimalkalkül*; this corresponds, in fact, to the *intended meaning* of these formalizations, viz. that of capturing – at a formal level – Bishop's point of view [**BCM**], not Brouwer's [**BI**].

and discards as incorrect ("non-strict") the intuitionistically unobjectionable *ex falso quodlibet* $[\bot \to \mathrm{A}]$.

On a proof-level, behind the "Rule of Clavius" $\neg\mathrm{A} \models \mathrm{A} \Rightarrow\ \models \mathrm{A}$, there is a *specific* inferential proof-operation [i.e., an abstraction operator] ε, say, which is definable classically, in $\lambda\gamma_{\&}\mathbf{CQ}$, by $\varepsilon \mathrm{x}{:}\neg\mathrm{A}.\mathrm{a}[\![\mathrm{x}]\!] \equiv_{df} \gamma\mathrm{x}{:}\neg\mathrm{A}.\mathrm{x}(\mathrm{a}[\![\mathrm{x}]\!])$. So, the type-theoretic variant of this – strange, old – proof-pattern [cf. Euclid **Elementa** IX.12, Christoph Klau (= Clavius) **Opera mathematica I**, Mayence 1611, *ad locum*, etc.] is:

$$\Gamma[\mathrm{x}{:}\neg\mathrm{A}] \vdash \mathrm{a}[\![\mathrm{x}]\!] : \mathrm{A} \Rightarrow \Gamma \vdash \varepsilon\mathrm{x}{:}\neg\mathrm{A}.\mathrm{a}[\![\mathrm{x}]\!] : \mathrm{A}.$$

Now, the ω's mentioned above [that is: the uses of *ex falso quodlibet*: $\omega_{\mathrm{A}}(\mathrm{e}) \equiv \gamma\mathrm{x}{:}\neg\mathrm{A}.\mathrm{e}$, with x not free in e], and the "Clavian" abstractions ε are "complementary" within/inside $\lambda\gamma_{\&}\mathbf{CQ}$, in the sense that they can be put together, in order to make up [the effect of] a use of *reductio ad absurdum* $[\gamma]$. On a provability level, this is well-known. Formally, one can define, in $\lambda\gamma_{\&}\mathbf{CQ}$,

$$\gamma^{\circ}\mathrm{x}{:}\neg\mathrm{A}.\mathrm{e}[\![\mathrm{x}]\!] \equiv_{df} \varepsilon\mathrm{x}{:}\neg\mathrm{A}.\omega_{\mathrm{A}}(\mathrm{e}[\![\mathrm{x}]\!]),$$

on a proof-(term)-level, and, in view of (a special case of) $[\oint\gamma]$, $\lambda\gamma_{\&}\mathbf{CQ}$ is able to identify the new γ°-abstraction with the old one. This can be generalized to a hierarchy of triples $[\gamma^{[n]},\varepsilon^{[n]},\omega^{[n]}]$, $(n \geq 0)$, that are collapsed back to $[\gamma,\varepsilon,\omega]$ by γ-diagonalization. Without $[\oint\gamma]$ or with assumptions weaker than $[\oint\gamma]$, one can distinguish a hierarchy of subsystems of $\lambda\gamma_{\&}\mathbf{CQ}$, that "prove the same theorems" [i.e., share the same *stratification criteria* for proof-terms], but are still equationally distinct (using different *identity-criteria*).

The addition of the ε-abstraction to the ordinary typed λ-calculus λ^{τ} yields the proof-theory of the inferential $[\bot,\to]$ part of **DQ**. The "positive" $[\to,\wedge,\forall]$ part of **DQ** is like in **CQ** and in *Minimalkalkül*, whereas the "negative" $[\vee,\exists]$-proof-operators of

DQ are slight generalizations of their *Minimalkalkül* and intuitionistic analogues. In the end, one can reconstruct proof-theoretically **DQ** as a typed λ-calculus, λ**DQ** say, that can be embedded into $\lambda\gamma_{\&}$**CQ**, as in the case of the Heyting proof-calculus. A complete description of λ**DQ** is tedious (although it has no "$\perp$-conversions" - since it has no ω's - it requires more "commuting conversions" than **HQ**): we give only the information that has been estimated useful in view of retrieving the basic ingredients.

7. *Post-completeness?* Internal evidence (as, e.g., among other things, the fact that $\lambda\gamma_{\&}$**CQ** is complete relative to the Heyting calculus, that it collapses many distinctions, like those among [γ,ε,ω]-triples, etc.) suggests the *conjecture* that $\lambda\gamma_{\&}$**CQ** is also *Post-complete*. That is: if **CQ** $\models$ A one cannot add a new closed equation $a_1 = a_2$, where a_1, a_2 are proofs of A, without loosing [Post-] consistency for the resulting extension. This is a typed analogue of a situation obtaining for the *extensional type-free* λ-*calculus*, where we cannot identify consistently two arbitrary normal forms [Corrado Böhm **Pubbl. IAC** (Rome) **696**, 1968]. Here, the normal[izabi]lity requirement is already insured by stratifiability.[9]

[9] In an *appendix*, we pay attention to *assorted topics*, somewhat loosely related to the main theme: the Gentzen *L*-systems, the concept of a proof-transformation for **CQ**, and the "double negation" interpretations of **CQ** into **HQ**, along the Kolmogorov (1925) translation-pattern.