AN ANCIENT LOGIC

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équivalences
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Het bekend zijn met de geschiedenies
van een bepaalde wetenschappelijke discipline
is een noodzakelijke stap voor het juiste begrip
van de huidige ontwikkelingen (*indien deze er zijn*),
terwijl het verleden alleen juist begrepen kan worden
door de juiste plaats te vouden binnen de huidige kennis.
(*Stelling* [Proposition] 8, in [96], June 4, 1981[1])

It is a pity that the pioneers of modern logic – also called "mathematical" or "symbolic" –, as, e.g. Boole, Frege, Peirce, Peano, Russell, etc., did not spend some time on the Stoic, mainly Chrysippean, fragments on logic that have survived: the effort could have been rewarding.[1]

What follows is a set of remarks on the conceptual structure of the logic of Chrysippus of Soli (cca 279 – cca 206), written from the point of view of modern proof-theory. There is a good reason to put "technical" comments before historical minutiae. Because, if we agree on the fact that Chrysippus was a reputed logician, as otherwise claimed by a longstanding tradition, then there are not too many distinct ways of saying, once more, in our terms, what he meant to say.

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[1]This dialectical (!) "theorem" [*Stelling*, in Dutch] was, actually, part of the Dissertation [95], Utrecht 1981. The Dutch quote is Henk Barendregt's version of my English original, which I lost. Here is a backwards translation, with some approximations, for the benefit of my Dutchless readers: "The fact of being familiar with the history of a specific scientific discipline is required in view of a correct understanding of the current developments – inasfar they exist –, while the past can be understood only by finding its right place within the contemporary knowledge." The (self-) quote illustrates best my way of understanding Gentzen's **LK** [45] via Chrysippus (sic). And conversely, perhaps, although the latter step is not necessarily yet another piece of *whig historiography*. — The remarks following below consist of a condensed summary of previous work (*Proof Structures in Traditional Logic* [1994–2007]: *An Ancient Logic* [2007]). The full talk, bearing the working title *Chrysippus and His Modern Readers* – half-stolen, *mutatis mutandis*, from Charles Lutwidge Dodgson (1832–1898), aka Lewis Carroll (cf. *Euclid and His Modern Rivals*, 1879, 1885[2], [30]) –, announced a while ago, is in draft-stage. [The remaining (foot)notes can be found at the end of the paper.]

§1 *Chrysippean logic*

The main claim of the present notes consists of saying that Chrysippus' logic was what we call, nowadays, "classical" logic. I shall first focus on the quantifier-free fragment of what I take to be "Chrysippus' logic".[2]

A preliminary observation: as a logician, Chrysippus payed attention to *logical form*, as opposed to mere *grammatical expression*. He was also concerned with the study of grammar, as well as with the relation between logical forms and their expression in natural language. On the other hand, he did not propose a formal, symbolic notation for logic constructs (the idea occurred to other people about twenty centuries later). Whence a good deal of Chrysippean – and, in general, Stoic – considerations on ambiguity and the like[3].

The basic Stoic (actually Chrysippean) logic concepts are: *proposition* (*axiōma*), *polar opposition* (or [logical] conflict), *entailment* (argument or even "syllogism", as a special case), and *rule(s) of inference* (*thema*(*ta*), more or less).

Propositions and entailments. For Chrysippus, the propositions (*axiōmata*) are abstract entities[4]; they can be either *simple* (atomic) or *complex*.

According to their meaning – "semantically" thus –, they fall into *opposita*, contradictory (better: "polar") pairs.[5]

In modern terms, a *Stoic argument* (*logos*, and oft also *sullogismos*, as a special case) is a finite sequence of propositions, where exactly one is tagged (as a *conclusion*). I shall use next *entailment* as a technical, neutral term, instead[6]. With this terminology, Stoic logic is an "entailment logic", not a "propositional" logic (*Satzlogik*), à la Frege (*Begriffsschrift*, 1879 [BS]), Peirce, Russell or Łukasiewicz.[7]

Formally, with $\Gamma := \mathrm{A}_1, ..., \mathrm{A}_n$, $(n > 0)$, an entailment can be written down as $\Gamma \vdash \mathrm{C}$ (the elements of Γ are called assumptions, *lēmmata*, in technical Chrysippean terminology), where the tag is the turnstile $\vdash$ itself, in guise of punctuation (read "therefore" or "yields"), and the conclusion C (*sumperasma*, or *epiphora*, in Stoic jargon) occurs last. Alternative reading: "C is a consequence of Γ".[8]

Like for the moderns, the main concern of logic, according to Chrysippus, consists of sorting out arguments (good vs bad): entailments can be valid or invalid, so that good arguments are expressed by valid entailments, bad arguments by invalid entailments.

Valid entailments can be generated from "axioms"[9], i.e., primitive valid entailments, called *indemonstrables* ([*logoi*] *anapodeiktoi*), by rules of inference (validity-preserving transitions). The logical move goes both ways, since by reversing the rules one can, ultimately, "reduce" any valid entailment to (valid) indemonstrable entailments – in guise of "axioms" –, in finitely many steps.[10] Implicitly, there is a claim of completeness behind the Stoic technique, because validity (for entailments) can be characterised, alternatively, by truth conditions for the corresponding conditionals, i.e., by a criterion of the form:

$$\text{the entailment } \Gamma \vdash \mathrm{C} \text{ is valid iff the conditional } (\&\Gamma \to \mathrm{C}) \text{ is true,}$$

where $\&\Gamma$ stands for the conjunction of the elements of Γ (taken in some "canonic" order, $(...(\mathrm{A}_1 \wedge \mathrm{A}_2) \wedge ... \wedge (\mathrm{A}_{n-1} \wedge \mathrm{A}_n)$, $n \geq 3$, by "associating to the left", say).[11]

Anyway, since one should not expect a very strict conceptual demarcation between (formal) syntax and semantics in the Stoic logical doctrines, the talk about the (would-be) "completeness of the Stoic system (of logic)" is a bit pointless and rather un-historical.

Rejections / refutations / (logical) conflict. In particular, a rejection / refutation (*elenchos*) is an entailment whose conclusion is a contradiction. This is rather tricky, because Chrysippean negation is not exactly a "primitive" idea, like in the moderns (Frege, Russell, Łukasiewicz etc.)

What is the (logical) form of the conclusion C, in the latter case? In other words, *how would a Stoic logician express a contradiction*?

Atomic propositions and the Chrysippean negation. The atomic propositions (atoms, for short) are, by definition, so to speak, divided into "polar" pairs. On the other hand, the atoms can be either "variable" or constants.

Proper atoms. The "variable" atoms, are taken as primitives. By way of example, one has polar pairs of the form: "It is day" *vs* "It is night", "Kallias is walking" *vs* "Kallias is sitting / standing"[12], or else, and better, in English "John is married" *vs* "John is a bachelor", or "n is odd" *vs* "n is even", for any particular $n > 0$, whence no real need for an "internal" negation, in order to express (proper) atomic polar oppositions. One might, indeed, think that the Stoic "atomic" negations make up a (semantic) feature of the natural language, they are not *indicators of logical form*.[13]

Atomic constants. In the case of propositional constants, one can pick up two arbitrary propositions, $\top$ (*verum*) resp. $\bot$ (*falsum*) say, whose truth values do not change according to the circumstances. *Examples:* $\top$:= "two is less than three" *vs* $\bot$:= "two is bigger than three".[14]

With this notation, a Greek rejection / refutation (*elenchos*) should be of the form $\Gamma \vdash \bot$, i.e., a rejection is an entailment where C [its conclusion] is an arbitrary false proposition (set C := $\bot$).

Formally, if A is an atom, let us write opp(A) for its polar opposite, and define the polar opposite of opp(A) as A := opp(opp(A)).[15] The "law of double negation" is, thus, "built-in, semantically", at atomic level, so to speak.[16]

Complex propositions. Complex propositions are built up, inductively, from simples (or atoms), by binary links (binary connectives), called "connectors" (*sundesmoi*). Formally, where # is a connector (*sundesmos*), (A # B) is a proposition, if so are its immediate components, A and B.[17]

The "method" of polar oppositions is used to classify complex propositions, as well. That is, complex propositions fall into pairs (A $\oplus$ B) vs (A $\otimes$ B), where $\oplus$ resp. $\otimes$ instantiate a specific connector # (see below).

Example. (A $\wedge$ B) *vs* (A $\vartriangle$ B), where $\wedge$ (**and**) is classical conjunction and $\vartriangle$ (**nand**) is its polar opposite (incompatibility, a "Sheffer functor").[18]

Like in atoms, we have (A $\wedge$ B) = opp(A $\vartriangle$ B), resp. (A $\vartriangle$ B) = opp(A $\wedge$ B) as meaning postulates for "opp", whence, again, "double negation": opp(opp(C) = C, for C := (A $\wedge$ B), resp. C := (A $\vartriangle$ B).

The procees is repeated for the remaining (polar) pairs. There are four of the kind left.

Tabulating as appropiate, in modern notation, the Stoic connectors amount to the following ten symbols, grouped in five polar teams (providing also due colloquial names in the meta-language):

[1] $\vartriangle$ (**nand**) vs $\wedge$ (**and**),

[2] $\rightarrow$ (**if**, material implication; reading approximatively: "if... then...") *vs* **more**, $\nrightarrow$ (its polar opposite: *mālon... ē...*, in Stoic parlance, a kind of "rather... than...", in English),

[3] $\leftarrow$ (**since**, co-implication, the converse of material implication; approximative reading: "since") *vs* **less**, $\nleftarrow$ (its polar opposite, in Stoic jargon: *ētton... ē...*, a kind of "rather not... than..."),

[4] $\vee$ (inclusive **or**) *vs* $\triangledown$ (its polar opposite, i.e. the analogous "Peirce-Sheffer functor" **nor**), and,

[5] $\leftrightarrow$ (**iff**, material equivalence) *vs* $\nleftrightarrow$ (**xor**, Boole's e**x**clusive **or**),

and similarly for the corresponding complex propositions.[19]

Let us call the first four [1]–[4] polar pairs (of connectors, resp. complex propositions) *proper* (polar pairs) and the latter [5] *sub-polar*, or *improper* (polar pairs).

One can easily see that the proper polars are well-behaved semantically: one has (classical) "disjunctions" on the left (LHS, in the above) and (classical) "conjunctions" on the right (RHS, above). The corresponding duals, in modern terminology, appear, in each case, in alternate pairs, while the sub-polars are self-dual.[20] In other words, the proper polars can be analysed back / decomposed into components, on a uniform pattern.

In particular, from this point of view, the first three Stoic indemonstrables [T1–T3] are different in character from the latter two [T4–T5].

Otherwise, the latter two can be eliminated definitionally: explicit definitions of **xor** – and thus **iff** – in terms of proper polars [and opp] can be found in the late Stoic textbook lore. Cf., e.g., Galen's *Inst. log.*, IV.3 – and the comments of John Sprangler Kieffer *ad loc.* –, for the definitional expansion of the Stoic exclusive disjunction [**xor**] in terms of [inclusive] **or** and **nand**, viz. $(A \nleftrightarrow B) =_{df} ((A \vee B) \wedge (A \vartriangle B))$ or, colloquially: "(A or B), but not (both A and B)".

On this subject, see also Bobzien 1999, p. 111, who reads correctly (i.e., truth-functionally) "A *māllon* B" and "A *ētton* B", resp. as "both (either A or B) and A" and "both (either A or B) and B", resp.[21], but, curiously, omits noticing explicitly the ("material") equivalences:

$((A \vartriangle B) \wedge A) \leftrightarrow (A \wedge \mathrm{opp}(B)) \leftrightarrow (A \nrightarrow B) \leftrightarrow \mathrm{opp}(A \rightarrow B)$

[= A *māllon ē* B, *scilicet*], resp.

$((A \vartriangle B) \wedge B) \leftrightarrow (\mathrm{opp}(A) \wedge B) \leftrightarrow (A \nleftarrow B) \leftrightarrow \mathrm{opp}(A \leftarrow B)$

[= A *ētton ē* B],

and, thereby, the polarity principle behind the Chrysippean ("semantic") construction.

On the other hand, the fact that the proper polars **if** and **nand** are interdefinable – if negation (resp. opp) is present – is to be handled separately. (See below.)

As to the proper polars, let us call, for convenience, the "disjunction"-like connectors ($\vartriangle$, $\rightarrow$, $\leftarrow$, $\vee$), appearing on the LHS, *additive*, and the "conjunction"-like connectors (appearing on the RHS), *multiplicative*, and similarly for the corresponding complex propositions. Generic notation: (A $\oplus$ B) resp. (A $\otimes$ B).

We have, again, semantically, for each pair ($\oplus$,$\otimes$), meaning postulates of the form:

$$\mathrm{opp}(A \oplus B) = A \otimes B, \text{ and } \mathrm{opp}(A \otimes B) = A \oplus B,$$

that is, in particular, (A $\vartriangle$ B) = opp(A $\wedge$ B) resp. (A $\wedge$ B) = opp(A $\vartriangle$ B), and so on, whence, in general,

$$C = \mathrm{opp}(\mathrm{opp}(C)), \text{ for each } C := (A \oplus B), \text{ resp. } C := (A \otimes B).$$

In other words, double negation is, again,"built-in", by construction.

One can understand the equivalences above as meaning postulates for "opp", whereby (classical) negation can be viewed as a defined notion, by setting, finally, non(A) := opp(A).

We can think of the above as a piece of semantics, which we can even formalise as appropriate. As already noted before, this does not mean that the Stoics would have cared to distinguish, conceptually, between bare syntax (as recent formalists would have it) and semantics (or model theory).[22]

Note also that the explanations above do not make any explicit appeal to a truth-value account of the Stoic connectors and of the concept of polar opposition. Whether this was actually the case in Chrysippus and his followers, we cannot tell with ultimate certainty, given the poor state of our sources. One can say, however, for sure, that the Stoics were well aware of the fact that most of the connectors they used to theorise explicitly upon can be characterised by something similar to our truth tables (as in Peirce, Frege, Russell, Post, and, later on, in Wittgenstein).

Once more, there is a close parallel to all this in contemporary proof-theoretic work concerning classical logic, mainly. Typically, in order to simplify the syntax of a Gentzen L-system, for instance, and to save repetitions, a proof-theorist would take the atoms to be as above, with only (classical) **and**

and (inclusive) **or** as proposition-forming primitives, defining next (classical) negation by the usual Ockham / de Morgan transformations. An equivalent technique consists of manipulating "signed" formulas, instead.[23] The least thing to say, here, is that the modern / contemporary techniques have been vastly anticipated by Chyrisppus and his followers.[24]

Analysis and polar projections. Next, for "analytical" purposes, so to speak, let us define *polar projections* left(C), right(C), for each complex proper polar proposition C, separately (leaving the sub-polars aside, for a while), as follows.[25]

For additive $\mathrm{C} := \mathrm{A} \oplus \mathrm{B}$:

If $\mathrm{C} = (\mathrm{A} \vartriangle \mathrm{B})$, then $\mathrm{left(C)} = \mathrm{opp(A)}$, $\mathrm{right(C)} = \mathrm{opp(B)}$.

If $\mathrm{C} = (\mathrm{A} \to \mathrm{B})$, then $\mathrm{left(C)} = \mathrm{opp(A)}$, $\mathrm{right(C)} = \mathrm{B}$.

If $\mathrm{C} = (\mathrm{A} \leftarrow \mathrm{B})$, then $\mathrm{left(C)} = \mathrm{A}$, $\mathrm{right(C)} = \mathrm{opp(B)}$.

If $\mathrm{C} = (\mathrm{A} \vee \mathrm{B})$, then $\mathrm{left(C)} = \mathrm{A}$, $\mathrm{right(C)} = \mathrm{B}$.

For multiplicative $\mathrm{C} := \mathrm{A} \otimes \mathrm{B}$:

If $\mathrm{C} = (\mathrm{A} \wedge \mathrm{B})$, then $\mathrm{left(C)} = \mathrm{A}$, $\mathrm{right(C)} = \mathrm{B}$.

If $\mathrm{C} = (\mathrm{A} \nrightarrow \mathrm{B})$, then $\mathrm{left(C)} = \mathrm{A}$, $\mathrm{right(C)} = \mathrm{opp(B)}$.

If $\mathrm{C} = (\mathrm{A} \nleftarrow \mathrm{B})$, then $\mathrm{left(C)} = \mathrm{opp(A)}$, $\mathrm{right(C)} = \mathrm{B}$.

If $\mathrm{C} = (\mathrm{A} \triangledown \mathrm{B})$, then $\mathrm{left(C)} = \mathrm{opp(A)}$, $\mathrm{right(C)} = \mathrm{opp(B)}$.

With this *schematic notation*, one has, as meaning postulates (semantically thus), equivalences of the form:

$$\mathrm{A} \oplus \mathrm{B} = \mathrm{left(A)} \vee \mathrm{right(B)},$$

$$\mathrm{A} \otimes \mathrm{B} = \mathrm{left(A)} \wedge \mathrm{right(B)},$$

i.e., the additives are (classical) disjunctions, while the multiplicatives are (classical) conjunctions, as announced already in the above.

Note that the usual (Boolean) duals are exactly those pairs that agree on polar projections. This is not the case for the sub-polar pair (**iff** vs **xor**), where each complex propositional form can be viewed either as an addivitve (classical disjunction) or as a multiplicative (classical conjunction). In other words, $(\mathrm{A} \leftrightarrow \mathrm{B})$ and $(\mathrm{A} \nleftrightarrow \mathrm{B})$ are self-duals.[26]

The Chrysippean logic as a rejection / refutation system. The fact that the overall construction described here corresponds actually to (what we call) classical logic, indeed, is obvious from the Stoic way of defining (valid) entailments (here: Stoic [valid] arguments).

First, we must remember the fact that entailments of the form Γ, A ⊢ C are to be analysed (actually: defined) in terms of rejections / refutations / (logical) conflict, by:

$$(\vdash)\ \Gamma,\ \mathrm{A} \vdash \mathrm{C} \Leftrightarrow \Gamma,\ \mathrm{A},\ \mathrm{opp(C)} \vdash \bot,$$

where Γ is as above, and ⇔ stands for equivalence (in the meta-language).

Here we may think of $\Gamma \vdash \bot$ as being a primitive monadic (single-place) predicate on sequences Γ. Ad hoc alternative notation: $\Gamma \models$, with intended reading, in modern terms: "Γ is inconsistent". So the above equivalence (⊢) can be wiewed as a definition of ⊢ in terms of ⊨, viz.

$$(\mathrm{df} \vdash)\ (\Gamma,\ \mathrm{A} \vdash \mathrm{C}) \Leftrightarrow_{df} (\Gamma,\ \mathrm{A},\ \mathrm{opp(C)} \vdash \bot)\ [\Leftrightarrow (\Gamma,\ \mathrm{A},\ \mathrm{opp(C)} \models)],$$

where Γ, taken as a parameter, might be empty, as a limit case.

In other words, the definition of a (valid) entailment would involve the (genuinely classical) *reductio ad absurdum*, as well.[27]

Explicitly, the basic tenet is that Chrysippean logic is constructed in terms of rejection / refutation (expressing [logical] conflict), taken as a primitive notion, to be further characterized "axiomatically" so to speak.

In modern terms, this covers an obvious induction, where defined is the (primitive recursive) monadic predicate ⊨.

With ⇒ standing for the meta-conditional, and & for conjunction in the meta-language, one has two "axioms":

$$(\bot \Vdash)\ \bot \models,$$

$$(\mathrm{cut} \Vdash)\ \mathrm{A},\ \mathrm{opp(A)} \models,$$

or even, more generally,

$$(\mathrm{cut} \Vdash \Gamma)\ \Gamma,\ \mathrm{A},\ \mathrm{opp(A)} \models,$$

and transitions (in Frege's terms [GGA]) or "structural" rules of inference (à la Gentzen 1934–1935) of the form:

(dil $\Vdash$) $(\Gamma \models) \Rightarrow (\Gamma, \mathrm{C} \models)$,

(prm $\Vdash$) $(\Gamma, \mathrm{A}, \mathrm{B} \models) \Rightarrow (\Gamma, \mathrm{B}, \mathrm{A} \models)$,

(CUT $\Vdash$) $(\Gamma, \mathrm{A} \models)$ & $(\Gamma, \mathrm{opp(A)} \models) \Rightarrow (\Gamma \models)$,

written in *parametric* form (i.e., keeping Γ as a parameter on both sides of $\Rightarrow$, where appropriate).

Note that a special case of the "inner" (cut $\Vdash$) is:

(cut $\Vdash \top$) $\top, \bot \models$

(see also below).

Here, the "axioms" make up the basis of the induction and the transitions the inductive step.[28]

The "inner"-cut axiom (cut $\Vdash$) is just a way of expressing the "law of (non-) contradiction".

Given the parametric spelling of the (primitive) rules above, from this, one has also, as a derived rule:

(ctc $\Vdash$) $(\Gamma, \mathrm{A}, \mathrm{A} \models) \Rightarrow (\Gamma, \mathrm{A} \models)$,

i.e., the so-called "contraction" rule (Frege's *Verschmelzung*, in GGA).

Here, (dil $\Vdash$), (prm $\Vdash$), (ctc $\Vdash$), and the global (CUT $\Vdash$)-rule stand for the usual "dilution", "permutation", "contraction" and the "syllogism" rules, resp., in Frege's GGA, as well as in Gentzen's *Inauguraldiss.* 1934–1935.[29]

Dilution and the fallacies of relevance. If we have at hand formal means in order to express the fact that a proof of C does not depend on Γ (ad hoc notation: $\mathrm{C}|\Gamma$), then (dil $\Vdash$) can be reversed, i.e., we have, also:

(dil $\Vdash \Gamma\Leftrightarrow$) $(\Gamma \models) \Leftrightarrow (\Gamma, \mathrm{A} \models)$, provided $\bot|\mathrm{A}$,

i.e., provided that the proof of the contradiction reached in the refutation ($\bot$, say) does not depend on A.[30]

Note that the reading of dilution (dil $\Vdash$) is quite natural, in this context, and does not involve would-be *fallacies of relevance*, as recent relevance logic defenders might imply, viz.:

> if (the sequence) Γ is inconsistent then, *a fortiori*, so is any proper extension of it[31].

The reversal of (dil-Γ): $(\Gamma, \mathrm{A} \models) \Leftarrow (\Gamma \models)$, provided $\perp|\mathrm{A}$, reads naturally, again:

> if (the sequence) (Γ, A) is inconsistent and A has never been used in establishing this very fact, then A is redundant and one can safely get rid of it, i.e., Γ is inconsistent, as well.

This way of understanding dilution (dil) and "redundancy" in Stoic logic contexts leaves to think that Chrysippus and his followers payed little or no attention to what the moderns would call *fallacies of relevance*.[32]

In particular, for $\mathrm{A} := \top$, one has

$$(\text{dil} \Vdash \top)\ (\Gamma, \top \models) \Rightarrow (\Gamma \models)$$

anyway, in view of the gobal ($\textsc{cut} \Vdash$)-rule, so that, with this stipulation, the "inner" (cut)-"axiom" makes redundant the "axiom":

$$(\perp \Vdash)\ \perp \models.$$

Actually, all "structural" rules (transitions), except the global ($\textsc{cut} \Vdash$), can be reversed, i.e., we have also:

$$(\text{prm} \Vdash \Leftrightarrow)\ (\Gamma, \mathrm{A}, \mathrm{B} \models) \Leftrightarrow (\Gamma, \mathrm{B}, \mathrm{A} \models),$$

$$(\text{ctc} \Vdash \Leftrightarrow)\ (\Gamma, \mathrm{A}, \mathrm{A} \models) \Leftrightarrow (\Gamma, \mathrm{A} \models).$$

The above, taken together with (df ⊢) generate the usual Frege-Gentzen ("structural") rules, as well as the "law of identity":

$$(\text{id})\ \mathrm{A} \vdash \mathrm{A},$$

resp. the "projection" rule:

$$(\text{id } \Gamma)\ \Gamma, \mathrm{A} \vdash \mathrm{A},$$

that can be also written as:

$$(\text{prj})\ \Gamma \vdash \mathrm{A}_i\ (0 < i < n+1)$$

(recall that Γ is a shorthand for $\mathrm{A}_1, ..., \mathrm{A}_n$).

A theōrēma dialektikon. Before leaving the subject, it might be a good idea to pause, once more, on Sextus Empiricus' statement of a so-called *dialectical theorem* [*theōrēma dialektikon*[33]] of the Stoics (Sextus *Adv. Math.*, VIII, 231, 3–6):

> "when we know the premisses [*lēmmata*] Γ which imply a certain conclusion [*sumperasma*] [C], we know also potentially the conclusion [C] involved in them [Γ] [*dunamei kakeīno en toutois echomen to sumperasma*], even though it is not explicitly [*kat' ekphoran*] stated" (Robert Gregg Bury's English, from the Loeb edition, *ad loc.*).
>
> As a side-remark, the Latin of the industrious Frenchman Gentien Hervet (1499–1584) is more readable: "*Cum habuerimus propositiones ex quibus colligitur aliqua conclusio, vi ac potestate in his habemus illam conclusionem, etiamsi diserte non enuntietur.*" (ed. Johann Albert Fabricius, Leipzig 1718, page 502).

For the Greekless reader, the quote contains a mix-up of Peripatetic and Stoic technical jargon (*sumperasma*, in place of *epiphora*, on a par with the *lēmmata*), so we may also wonder where was actually Sextus copying from.

In the light of the above, the "dialectical theorem" referred to by Sextus looks, rather, like an inductive stipulation.

Take first Γ modulo arbitrary permutations (i.e., as a multiset; this is implicit in the Chrysippean way of understanding the *lēmmata*, anyway).

Then define $\Gamma \succ C$ [i.e., C *dunamei en tois* Γ], inductively, by:

> [1] C is an element of Γ [basis clause for $(\Gamma \succ C)$],
>
> [2] there is a proposition A[34] such that $\Gamma, A \succ C$, and $\Gamma \succ A$ [inductive step, with same conculsion $(\Gamma \succ C)$, via the global (CUT)].

The basis clause (of the induction) [1] is a diluted (id) – i.e. the "projection axiom" (prj) above –, covering (dil), as well, while the inductive step [2] covers the remaining "structural" rules of inference, viz. (CUT) and (ctc) [as well as (prm), in fact, given the multiset assumption on Γ's].

If the over-argutious Sextus was actually quoting Chrysippus or a genuine Stoic source[35], one can only admire this concise intuitive phrasing of a basical logical idea that took about twenty two centuries to be retrieved.[36]

Proper logical rules. What about the proper "logical" rules (transitions), i.e., those involving connectors?

With the projective shorthand-notation above, one has, for each polar pair $(\oplus,\otimes)$, equivalences of the form:

[$\otimes$] multiplicative case, where C := (A $\otimes$ B),

$(\otimes)$ $(\Gamma, \text{left(C)}, \text{right(C)} \models) \Leftrightarrow (\Gamma, \text{C} \models)$,

[$\oplus$] additive case, where C := (A $\oplus$ B),

$(\oplus)$ $(\Gamma, \text{opp(left(C))} \models)$ & $(\Gamma, \text{opp(right(C))} \models) \Leftrightarrow (\Gamma, \text{C} \models)$,

where the left-to-right transition is a corresponding Gentzen-rule (an "introduction" rule), while a right-to-left transition is the associated resolution-rule (as in the Beth-Hintikka-Smullyan tableaux, etc.; an "elimination" rule, thus).

Example. Case of this, taking $\otimes$ to be $\wedge$ and $\oplus$ to be $\vartriangle$:

$(\wedge)$ $(\Gamma, \text{A}, \text{B} \models) \Leftrightarrow (\Gamma, (\text{A} \wedge \text{B}) \models)$,

$(\vartriangle)$ $(\Gamma, \text{opp(A)} \models)$ & $(\Gamma, \text{opp(B)} \models) \Leftrightarrow (\Gamma, (\text{A} \vartriangle \text{B}) \models)$,

and, since one has defined not(A) := opp(A), in the latter case, one has:

$(\vartriangle)$ $(\Gamma, \text{not(A)} \models)$ & $(\Gamma, \text{not(B)} \models) \Leftrightarrow (\Gamma, (\text{A} \vartriangle \text{B}) \models)$.

Putting (df $\vdash$) at work yields:

$(\wedge)$ $(\Gamma, \text{A}, \text{B} \vdash \text{C}) \Leftrightarrow (\Gamma\ (\text{A} \wedge \text{B}) \vdash \text{C})$ ["confusion"]

$(\vartriangle)$ $(\Gamma, (\text{not(A)} \vdash \text{C})$ & $(\Gamma, \text{not(B)} \vdash \text{C}) \Leftrightarrow (\Gamma, (\text{A} \vartriangle \text{B}) \vdash \text{C})$,

with also, in view of (df $\vdash$), derived rules:

(adj) $(\Gamma \vdash \text{A})$ & $(\Gamma \vdash \text{B}) \Rightarrow (\Gamma \vdash (\text{A} \wedge \text{B}))$

[the "adjunction rule" for $\wedge$],

as well as the expected $\wedge$-"projections", i.e., the "simplification" or "elimination" rules for $\wedge$:

(fst $\wedge$) $(\Gamma \vdash (\text{A} \wedge \text{B})) \Rightarrow (\Gamma \vdash \text{A})$,

(snd $\wedge$) $(\Gamma \vdash (\text{A} \wedge \text{B})) \Rightarrow (\Gamma \vdash \text{B})$.

Note that the ($\triangle$,$\wedge$)-team above generates also the derived rule ($\triangle$-introduction on the right):

(split) (Γ, A, B $\vdash \bot$) $\Rightarrow$ ($\Gamma \vdash$ A $\triangle$ B).

On the other hand, the meaning postulates for the built-in negation, yield derived equivalences of the kind:

(Γ, A $\vdash \bot$) $\Leftrightarrow$ ($\Gamma \vdash$ not(A)),
(Γ, not(A) $\vdash \bot$) $\Leftrightarrow$ ($\Gamma \vdash$ A),
(Γ, not(not(A)) $\vdash$ C) $\Leftrightarrow$ (Γ, A $\vdash$ C) $\Leftrightarrow$ (Γ, A $\vdash$ not(not(C))), etc.

In the end, by the standards above, a Gentzen sequent "multiple on the right" (i.e., something of the form $\Gamma_1 \vdash \Gamma_2$, where Γ_i (i := 1,2) are finite sequences of propositions) in just a Stoic rejection / refutation (i.e., of the form: Γ_1, opp(Γ_2) $\vdash \bot$, where opp(Γ_2) := opp(B_1), ..., opp(B_m), for Γ_2 := B_1, ..., B_m, $m > 1$).[37]

All this is very redundant, of course. As a matter of fact, a single (proper) polar pair ($\oplus$,$\otimes$) is sufficient in order to get full classical (propositional) logic.

Chrysippus' logic is classical logic. Let us call, for further reference, **Ch** ["Ch" for **Ch**riyppus] this (global) formulation of classical logic. More precisely, we may refer to it as **Ch**[$\Vdash$], in order to stress the fact that the rejection predicate [$\Vdash$] has been taken as a (semantical) primitive. The proof that **Ch** is classical (propositional) logic, indeed, is straightforward. [*Hint.* Construct, first, an algebra **Ch-seq**, say, on finite sequences of propositions, satisfying the "Chrysippean" conditions stated earlier. Show next that **Ch-seq** is a Boolean algebra [BA]. Finally, concoct a Stone-like argument to the effect that every BA can be so represented. Whence "classical completeness".[38]]

Modulo reserves already alluded to in the above, regarding the distinction syntax *vs* semantics in Ancient logic, **Ch** might be thought of as being a kind of genuine "semantical" justification of (what we understand, nowadays, by) classical logic (as opposed to intuitionistic logic, say, based on the so-caled BHK [Brouwer-Heyting-Kolmogorov] interpretation of the Brouwer-Heyting logic).[39] Note also that the explanations above did not make appeal explicitly to a truth-functional account of the Chrysippean connectors and of the concept of polar opposition.[40] Incidentally, one can also realise the fact that, once valid entailments are characterised on the proposed pattern, the predicates **True** and **False** – as applied to propositions – can be defined explicitly in terms of monadic entailments, by **True**(A) iff $\top \vdash$ A resp. **False**(A) iff A $\vdash \bot$. (See also the previous remarks on classical completeness.)

§2 *Redundancies and "semantic" fibrations*

Summing up, given (df $\vdash$), one can easily derive, from **Ch**[$\Vdash$], the following "structural" bi-transitions, in terms of $\vdash$ alone:

($\vdash$ dil $\Leftrightarrow$) $\Gamma \vdash C \Leftrightarrow \Gamma, A \vdash C$, provided C does not depend on A,

($\vdash$ prm $\Leftrightarrow$) $\Gamma, A, B \vdash C \Leftrightarrow \Gamma, B, A \vdash C$,

($\vdash$ ctc $\Leftrightarrow$) $\Gamma, A, A \vdash C \Leftrightarrow \Gamma, A \vdash C$,

together with the following forms of global (CUT):

(CUT $\vdash \otimes$) $(\Gamma, A \vdash C)$ & $(\Gamma \vdash A) \Rightarrow (\Gamma \vdash C\ )$ ["parametric"],

(CUT $\vdash \oplus$) $(\Gamma_1, A \vdash C)$ & $(\Gamma_2 \vdash A) \Rightarrow (\Gamma_1, \Gamma_2 \vdash C)$ ["cumulative"].

As already mentioned above, the "inner" (cut)-"axiom" (the "law of (non)-contradiction") is tantamount "the law of identity":

(id) $A \vdash A$,

in this setting, whence the "projection" rule:

(prj) $\Gamma, A \vdash A$

follows by (repeated) dilutions.

On the other hand, in the case of the polar pair of connectors [$\wedge$,$\vartriangle$], the proper "logical" rules of inference (those involving connectors), yield:

($\vdash \wedge \Leftrightarrow$) $(\Gamma, A, B \vdash C) \Leftrightarrow (\Gamma, (A \wedge B) \vdash C)$ ["confusion"],

($\vdash \vartriangle \Leftrightarrow$) $(\Gamma, \mathrm{opp}(A) \vdash C)$ & $(\Gamma, \mathrm{opp}(B) \vdash C) \Leftrightarrow \Gamma, (A \vartriangle B) \vdash C$,

whereas, from the latter, we get, as noted before:

$(\vartriangle \vdash \Leftrightarrow)$ $(\Gamma,\ \mathrm{A},\ \mathrm{B} \vdash \bot) \Leftrightarrow (\Gamma \vdash (\mathrm{A} \vartriangle \mathrm{B}))$,

$(\wedge \vdash \Leftrightarrow)$ $(\Gamma \vdash \mathrm{A})$ & $(\Gamma \vdash \mathrm{B}) \Leftrightarrow (\Gamma \vdash (\mathrm{A} \wedge \mathrm{B}))$.

i.e., the "split"-rule ($\vartriangle$-"introduction" on the right), resp. the usual "adjunction"- and "projections"-rules for $\wedge$.

The other three polar pairs of connectors yield the expected bi-transitions (in terms of $\vdash$).

In particular the $[\nrightarrow,\rightarrow]$-pair gives:

$(\vdash \nrightarrow \Leftrightarrow)$ $(\Gamma,\ \mathrm{A},\ \mathrm{opp}(\mathrm{B}) \vdash \mathrm{C}) \Leftrightarrow (\Gamma,\ (\mathrm{A} \nrightarrow \mathrm{B}) \vdash \mathrm{C})$

[a case of "confusion"],

$(\vdash \rightarrow \Leftrightarrow)$ $(\Gamma,\ \mathrm{opp}(\mathrm{A}) \vdash \mathrm{C})$ & $(\Gamma,\ \mathrm{B} \vdash \mathrm{C}) \Leftrightarrow (\Gamma,\ (\mathrm{A} \rightarrow \mathrm{B}) \vdash \mathrm{C})$,

whereas the latter two yield (in this order):

$(\rightarrow \vdash \Leftrightarrow)$ $(\Gamma,\ \mathrm{A} \vdash \mathrm{B}) \Leftrightarrow (\Gamma \vdash (\mathrm{A} \rightarrow \mathrm{B}))$,

$(\nrightarrow \vdash \Leftrightarrow)$ $(\Gamma \vdash \mathrm{A})$ & $(\Gamma,\ \mathrm{B} \vdash \bot) \Leftrightarrow (\Gamma \vdash (\mathrm{A} \nrightarrow \mathrm{B}))$, resp.

$(\nrightarrow \vdash \Leftrightarrow)$ $(\Gamma \vdash \mathrm{A})$ & $(\Gamma \vdash \mathrm{opp}(\mathrm{B})) \Leftrightarrow (\Gamma \vdash (\mathrm{A} \nrightarrow \mathrm{B}))$.

An easy excercise shows that the definitional stipulations (equivalences):

$$(\mathrm{A} \rightarrow \mathrm{B}) = (\mathrm{A} \vartriangle \mathrm{opp}(\mathrm{B})),$$
$$(\mathrm{A} \nrightarrow \mathrm{B}) = (\mathrm{A} \wedge \mathrm{opp}(\mathrm{B}))$$

allow the derivation of the $[\nrightarrow,\rightarrow]$-equivalences from the corresponding $[\wedge,\vartriangle]$-equivalences, listed previously, whereas the (definitional) stipulations:

$$(\mathrm{A} \vartriangle \mathrm{B}) = (\mathrm{A} \rightarrow \mathrm{opp}(\mathrm{B})),$$
$$(\mathrm{A} \wedge \mathrm{B}) = (\mathrm{A} \nrightarrow \mathrm{opp}(\mathrm{B})) = \mathrm{opp}(\mathrm{A} \rightarrow \mathrm{opp}(\mathrm{B}))$$

guarante the corresponding derivations in the opposite direction.

As expected, the remaining proper polar pairs ($[\nleftarrow,\leftarrow]$, resp. $[\triangledown,\vee]$) yield analogous equivalences (bi-transitions) in terms of $\vdash$ alone, whereas the appropriate definitional equivalences allow the derivation of the latter teams from either one of those mentioned previously, and, ultimately, from those generated by the $[\wedge,\vartriangle]$-pair alone, for instance.

Let us call the overall [re-] construction of **Ch**[$\Vdash$] in terms of $\vdash$, **Ch**[$\vdash$]. It is obvious that the latter formulation (in terms of entailments) is equivalent with the former one (in terms of refutations)[41]. In other words, both "semantic" formulations – **Ch**[$\Vdash$] and **Ch**[$\vdash$] – turn out to be equivalent, in the sense they allow deriving the same set of rules [transitions], modulo (df $\vdash$). Moreover, the discussion above shows that we can eventually fibrate **Ch** "semantically" along a single proper polar pair of connectors, i.e., cut off would-be redundancies, on this pattern, in four distinct ways.

From a *genuinely Chrysippean* point of view, the most interesting and (historically) relevant "semantic" fibration appears to be the one along the [$\wedge$,$\triangle$] polar pair of connectors.[42]

A historical aside. One might argue that Chrysippus could have borrowed the so-called "hypothetical syllogisms" involving conditionals and exclusive disjunctions (as mentioned explicitly in the Stoic indemonstrables T1–T2, and T4–T5, resp.) from late Peripatetic sources (Theophrastus, Eudemus, and "some other of Aristotle's associates [*hetairoi*]", as Alexander of Aphrodisias has occasionally implied[43]). Putting aside the – quite obvious – fact that neither the Great Aristotle nor his lesser *hetairoi* – or later (Peripatetic) followers, for that matter – had the slightest idea about what is a logical connector [proposition-forming binary connective][44], even the astute, bright and partinic (!) connoisseur Alexander did not attempt to assign the discovery of the **nand**-connector to any one of his fellow (Peripatetic) predecessors who lived in the shadow of the Great Master.[45]

What follows in the next section concerns "syntactic" constructions, based on (the intented semantics of) **Ch**, whose association to the actual Stoic ideas are, historically speaking, rather conjectural. They are, however, in the Chrysippean spirit, so to speak.

§3 *"Syntactic" fibrations*

As noted above, in view of the well-known interdefinability of classical connectives, **Ch** is very redundant. Like, *mutatis mutandis*, Gentzen's **LK** – the sequent-version of classical logic, in his 1934–1935 –, actually.[46] Except for the fact that, unlike in Gentzen's **LK**, the "Ancient Logic" **Ch** – as presented here – has an implicit *conceptual justification*, as well as an explicit *criterion of construction* (which, otherwise, Gentzen missed).

It is instructive to examine subsystems or fragments of **Ch**, say, based on functionally complete sets of classical (Boolean) connectives.

Leaving the sub-polar pair (**iff**, **xor**) aside, one can *prima facie* fibrate "semantically" the overall construction [**Ch**], in four distinct ways, as suggested above, by choosing a single specific (proper) polar pair of connectors ($\oplus$,$\otimes$), as a primitive setting, while still tinkering on (the status of) negation.

Further, one can specialise this choice, by taking only one of the connectors, (classical) negation, and a constant ($\top$ or $\bot$) as primitives. This yields eight possible (syntactic) fibrations of **Ch**, so to speak.

In two cases (those involving **nand** resp. **nor** as primitives), negation is redundant (as a primitive), because we have:

$$\text{not(A)} = (\text{A} \vartriangle \text{A}) = (\text{A} \triangledown \text{A}),$$

as meaning postulates.

In these cases, redundancy is "built-in" so to speak, since at least one propositional constant is necessary for functional (Boolean) completenesss, while, granted $\bot$, say, we can define, alternatively, an inferential negation (à la Peirce 1885), by setting:

$$\text{not(A)} := (\text{A} \to \bot)$$

(as material implication is definable in terms of $\vartriangle$ resp. $\triangledown$ alone, without using propositional constants), etc.

On the other hand, the other two cases (involving $\wedge$ resp. $\vee$ as primitives), one needs a primitive negation, as well, whence again, "built-in" redundancy, since one needs at least one constant, as above, in order to guarantee functional completeness.

Now, in the remaining four cases (where either $\to$ or its converse $\leftarrow$ is present) the only non-redundant choices consist of taking:

(1) either [$\bot$,$\to$] resp. [$\bot$,$\leftarrow$] as primitive signatures, with negation defined "inferentially", by:

$$\text{not(A)} := (\text{A} \to \bot)\text{, as above, or by:}$$

$$\text{not(A)} := (\bot \leftarrow \text{A})\text{, resp.}$$

(2) or [$\top$,$\nrightarrow$] resp. [$\top$,$\nleftarrow$] as primitive signatures, with negation defined "co-inferentially", by:

$\mathrm{not}(A) := (\top \nrightarrow A)$, or by:

$\mathrm{not}(A) := (A \nleftarrow \top)$,

and it is relatively easy to see that any other one of the remaining choices has "built-in" redundancy, too.

Some reflection on this elementary combinatorics shows that the most economic – and conceptually clean – choices of primitives are those involving an additive connector ($\vartriangle$, $\rightarrow$, $\leftarrow$, $\vee$).

On the other hand, it is obvious that there is no (conceptual) profit in using $\leftarrow$ instead of $\rightarrow$, resp. $\vee$ instead of $\vartriangle$, as primitives (the latter behave, proof-theoretically, exactly in the same way, modulo trivial transformations: $(A \leftarrow B) = (B \rightarrow A)$, on the one hand, resp. $(A \vee B) = (\mathrm{not}(A) \vartriangle \mathrm{not}(B))$, by Ockham / de Morgan, on the other hand), so that we end up with just two, conceptually distinct, non-redundant strategies, viz. with the primitive signatures $[\bot,\vartriangle]$ and $[\bot,\mathrm{not},\rightarrow]$, resp. This yields, essentially, two conceptually distinct ways of formulating (syntactically) the "Ancient Logic" of Chrysippus.

An Ancient Logic, Formulation 1: the $[\bot,\vartriangle]$*-case.* Let us introduce, first, the fragment $\mathbf{Ch}[\bot,\vartriangle]$ of $\mathbf{Ch}$, based on the primitive (propositional) "signature" $[\bot,\vartriangle]$ alone.

Taking $\vartriangle$ as a primitive, together with $\bot$, allows defining the remaining classical connectives (sic – including thus classical negation, understood as a [modern] connective, this time) on a familiar, well-known pattern, by setting (definitionally), e.g.:

$\mathrm{not}(A) := (A \vartriangle A)$,

$(A \wedge B) := \mathrm{not}(A \vartriangle B)$,

$(A \rightarrow B) := (A \vartriangle \mathrm{not}(B))$,

$(A \leftarrow B) := (\mathrm{not}(A) \vartriangle B)$,

$(A \vee B) := (\mathrm{not}(A) \vartriangle \mathrm{not}(B))$,

etc., so that, for instance, on the primitive signature above, granted the appropriate team of "structural" rules for $\vdash$ (including the global (CUT) for $\vdash$), the following rules of inference are sufficient for full classical (propositional) logic:

(i-cut) $(\Gamma \vdash \text{non(A)})$ & $(\Gamma \vdash \text{A}) \Rightarrow (\Gamma \vdash \bot)$

["the law of (non-) contradiction"],

(red) $(\Gamma, \text{non(A)} \vdash \bot) \Rightarrow (\Gamma \vdash \text{A})$ [*reductio ad absurdum*],

(fst) $(\Gamma \vdash \text{A} \wedge \text{B}) \Rightarrow (\Gamma \vdash \text{A})$,

(snd) $(\Gamma \vdash \text{A} \wedge \text{B}) \Rightarrow (\Gamma \vdash \text{B})$,

(adj) $\Gamma\ (\vdash \text{A})$ & $(\vdash \text{B}) \Rightarrow \Gamma \vdash (\text{A} \wedge \text{B})$ ["adjunction"],

Remember that, in this setting, conjunction $[\wedge]$ is a defined notion, i.e., $(\text{A} \wedge \text{B}) := \text{not}(\text{A} \vartriangle \text{B})$. Here, we just ignore the (two) remaining ways of defining negation ("inferentially").

An Ancient Logic, Formulation 2: the $[\bot,\text{not},\rightarrow]$*-case.* The most preferred modern formulation of classical logic (Frege, Church, Łukasiewicz, Jaśkowski etc.) relies on a primitive $[\text{not},\rightarrow]$-signature. The latter is, actually, a functionally incomplete set, since the propositional constants are missing.[47]

On the other hand, adding a primitive propositional constant (pick up $\bot$, for instance) yields "built-in" redundancy as noted before, in view of the inferential definition(s) of negation, à la Peirce. Nevertheless, as in the case of Formulation 1 above, there is no reason to bother about, since one can, simply, ignore the inferential alternative(s).

The $[\bot,\text{not},\rightarrow]$-fragment of **Ch**, $\mathbf{Ch}[\bot,\text{not},\rightarrow]$ say, consists of the following rules of inference:

(i-cut) $(\Gamma \vdash \text{not(A)})$ & $(\Gamma \vdash \text{A}) \Rightarrow \Gamma \vdash \bot$

["the law of (non-) contradiction"],

(red) $\Gamma, \text{not(A)} \vdash \bot \Rightarrow \Gamma \vdash \text{A}$

[*reductio ad absurdum*],

exactly as for $\mathbf{Ch}[\bot,\vartriangle]$ above, meant to handle the proof-theoretical behaviour of negation (and $\bot$), with, moreover, a "replication" of the (i-cut)-(red)-team:

($\rightarrow$-cut) $(\Gamma \vdash \text{A} \rightarrow \text{B})$ & $(\Gamma \vdash \text{A}) \Rightarrow \Gamma \vdash \text{B}$

[*modus ponens*, $\rightarrow$-elimination]

(abs) $\Gamma, \text{A} \vdash \text{B} \Rightarrow \Gamma \vdash \text{A} \rightarrow \text{B}$

["the deduction theorem", $\rightarrow$-introduction], meant to handle the proof-theoretic behaviour of (minimal, as well as material) implication.

§4 *Closing remarks*

Some variations and / or improvements on the "syntactic" fibrations mentioned earlier are still possible.

Inferential Formulations. From Formulation 2, Ch[$\bot$,not,$\to$], we can get a non-redundant Formulation 2i, **Ch**[$\bot$,$\to$] – with "i" short for "inferential" –, by defining not(A) "inferentially", as above, whereby (i-cut) becomes a special case of ($\to$-cut) = (*modus ponens*).

Double negation. On the other hand, from both Formulation 1 and 2, one can obtain slightly redundant Formulations 1[DN] and 2[DN] say, on the same primitive propositional signatures, by adding two double-negation [DN] rules, that are actually redundant in both Formulation 1 and Formulation 2 (resp. 2i),

$(\nabla)\ \Gamma \vdash \mathrm{A} \Rightarrow \Gamma \vdash \mathrm{not(not(A))}$ [double-negation introduction],

$(\Delta)\ \Gamma \vdash \mathrm{not(not(A))} \Rightarrow \Gamma \vdash \mathrm{A}$ [double-negation elimination].

The latter Formulations are significantly more efficient, in practice, as well as much cleaner, conceptually.

As a matter of fact, the conceptual profit of a DN-formulation is visible only if we are interested in a "witness theoretic" presentation of classical logic, by also "formalizing the proofs themselves", so to speak. In the DN-cases, the "witnessed" DN-rules are supposed to obbey obvious inversion-principles, governing witnesses / proofs, of the form:

$[\nabla]\ \Gamma \vdash \nabla(\mathrm{a}) : \mathrm{not(not(A))}$, if $\Gamma \vdash \mathrm{a} : \mathrm{A}$,

$[\Delta]\ \Gamma \vdash \Delta(\mathrm{c}) : \mathrm{A}$, if $\Gamma \vdash \mathrm{c} : \mathrm{not(not(A))}$,

for all witnesses a : A, resp. c : not(not(A)) relative to Γ, subjected to explicit (witness / proof) isomorphisms making up an inversion (as in groups, say):

$(\beta\Delta) \vdash \nabla(\Delta(\mathrm{c})) = \mathrm{c} : \mathrm{not(not(A))}$,

$(\eta\Delta) \vdash \Delta(\nabla(\mathrm{a})) = \mathrm{a} : \mathrm{A}$, resp. (where = stands for proof-conversion or proof-isomorphism, this time). It is easy to see that in the DN-less formulations at least one of the $(\beta/\eta\Delta)$-conditions would normally fail (in the appropriate λ-calculus), whence the idea of taking ∇ and Δ as primitive (proof-) operators (resp. rules of inference).

Witness theories. Roughly speaking, the "witness theories" corresponding to the above are as follows:

> Formulation 1 **Ch**[$\perp$,$\triangle$] coresponds to the $\lambda\pi$-calculus $\boldsymbol{\lambda\pi}$ (λ-calculus with "surjective pairing"), typed as appropriate (Rezuş 1993a).
>
> Formulation 2i **Ch**[$\perp$,$\rightarrow$] corresponds to typed $\lambda\gamma$-calculus (Rezuş, cca 1987). Cf., e.g., Rezuş 1990, 1991, 1993, building upon the pioneering work of Dag Prawitz, PhD Diss. Stockholm 1965. A similar λ-calculus-based construction ($\lambda\mu$-calculus) has been proposed by Michel Parigot, around 1991.
>
> Formulation 2 **Ch**[$\perp$,not,$\rightarrow$] and the "redundant" DN-formulations are variations on Rezuş 1993a. The corresponding (typed) λ-calculi are replications [here, just duplications] of pure λ-calculus and, actually, proper subsystems of $\lambda\pi$-calculus, even at undecorated ["type-free"] level, since, unlike the pure λ-calculus, $\boldsymbol{\lambda\pi}$ contains infinitely many distinct, notrivial copies of itself.

There are many more such, but the [sub-] systems listed in the above are very close to the original Stoic **Ch** system, and to the "Chrysippean" way of justifying classical logic.

§5 *Bibliographical notes*

As regards the relevant textual sources, the edition of Hans van Arnim 1903–1905, 1924 [SVF] is also vailable in recent reprints. It has been partially translated in several modern languages (see, e.g., Dufour 2004 [French], Baldassari 1984, and Radice 1989 [Italian]). The new, comprehensive collection of Hülser 1987–1988 contains also a German translation (otherwise not always very inspired). Although included in the fragment-collections listed above, other specific sources (Alexander of Aphrodisias, Sextus Empiricus, and Galenus) have been listed separately.

Pace Charles S. Peirce, the only (more or less professional) mathematician referring explicitly to Chrysippus, I know of, is Gerolamo Cardano (1501–1576); cf. Cardano 1570, 1663, and, possibly, Rezuş 1991, Bellissima & Pagli 1996.

For the "traditional" views on Stoic logic, see Prantl 1855 (vol. 1) and Zeller 1879 (German original), 1892 (English translation).

For modern technical discussions and / or "reconstructions" of Chrysippus' logic, see Łukasiewicz 1934, Mates 1948, 1953, Becker 1957, Kneale & Kneale 1962; Egli 1967, 1978, 1979, 1993, 2000 (mainly, on "Stoic quantifiers"), Frede 1974, Gould 1974, Corcoran 1974, Mueller 1974, 1978, 1979, Brunschwig 1980, Ierodiakonou 1990, Mignucci 1993, Bobzien 1996, 1999, 2003, 2016, 2016a, Nasieniewski 1998, O'Toole & Jennings 2004 (this item contains too many many errors to be useable; not only tyographical, unfortunately), and, last but not least, the John Locke Oxford Lectures (Summer 2004) of Jonathan Barnes 2007. (See also Barnes 1980, 1985, 1996, 1999.)

On Frege's BS (1879) *vs* GGA (1893) – i.e., his *Regellogik* –, see also Rezuş 2009 (rev. 2016). For Charles S. Peirce, see Peirce 1880, 1902 (as a precursor of Sheffer 1913) and Peirce 1885 (for the definition of "inferential" negation, etc.). For details on Henry M. Sheffer, see Scanlan 2000, and Urquhart 2012.

For the origins of so-called natural deduction and sequent logic, and for recent technicalities on the subject, see Hertz 1922, 1923, 1928, 1929,1929a, Legris' (2012) introduction to his translation of Hertz 1922, Jaśkowski 1927, 1934 (work of 1926–1927), Gentzen 1932 (on Hertz), and his *Inauguraldiss.* 1934–1935, Fitch 1952, Prawitz 1965, 1971, 1973, 1974, 1977, 1979, 1981, Rezuş 1981, 1990, 1991, 1993, Indrzejczak 1998, 2016, Pelletier 1999, 2001, Barendregt & Ghilezan 2000, Hazen & Pelletier 2012, 2014, and so on. (On the behaviour of the "Peirce-Sheffer functors" – i.e., **nand** and **nor** – in this context, cf., e.g., Price 1961, von Kutschera 1962, Gagnon 1976, Read 1999, Zach 2015, etc.)

For λ-calculus and type-theories based on λ-calculus, see Barendregt 1981 (second, revised edition: 1984), 1992, Rezuş 1981, 1986, 1986a, Hindley & Seldin 1986, 2008, Hindley 1997, Barendregt *et al.* 2013, and, possibly, Rezuş 2015 (a review of the latter item). On the Curry-Howard Correspondence ("proposition as types") for classical logic, see Rezuş 1990, 1991, 1993, 1993a and Sørensen & Urzyczyn 2006 (containing also a brief description of

Parigot's $\lambda\mu$-calculus, mentioned in the above). Nicolaas G. de Bruijn's AUTOMATH proof-systems are documented in de Bruijn 1980, Rezuş 1983, and Barendregt & Rezuş 1983.

For modern "polar" proof-theoretical / semantical constructions, see Novikov 1941, Schütte 1977 and the references – to Girard – appearing in Girard's Rome 2004-lectures, issued now also in English, as Girard 2011.

Finally, the early history of the so-called tableaux-systems [Beth-Hintikka] can be recovered from the Amsterdam PhD Dissertation of Paul van Ulsen 2000, while the very basics on tableaux can be retrieved from the monograph Smullyan 1968 and the survey of D'Agostino 1999. Readers interested in the computer-science counterpart of the same story (resolution) might profit from perusing the booklet of John Alan Robinson (Robinson 1979), the father of resolution.

Acknowledgements

I am indebted to Susanne Bobzien (All Souls, University of Oxford) for prompting me coming back to one of my preferred spare time subjects (viz. the logic of Chrysippus), and, *lato sensu*, to my friend and (private) teacher of Ancient Greek, Petru Creţia (1927–1997), the editor of Plato in Romanian.

Notes

[1]They might have had a historical excuse: the comprehensive edition of Hans von Arnim (*Stoicorum Veterum Fragmenta* [SVF]) was published by the turn of the previous century [three volumes, 1903–1905, the fourth one, containing Adler's index, is dated 1924], von Arnim's edition was incomplete inasafar logic was concerned, the received views (Carl [von] Prantl, Eduard Zeller *et al.*) on Stoic logic were rather innacurate – to say the least –, while the first competent person to realise the logic significance of the Stoic corpus, Jan Łukasiewicz (circa 1923), managed to publish an account of his findings only in 1934. Relevant studies of Stoic logic re-emerged, in the footsteps of Łukasiewicz, only after the WWII, during the late forties and the fifties (Benson Mates and Oskar Becker; cf. Mates' UCB PhD Dissertation 1948, published in 1953, and Becker's notes *Über die vier Themata der stoischen Logik*, 1957.) So far, we have about a dozen – or so – of "technical" studies in print, on the subject, worth mentioning. On this line of research, most authors have been involved in "reconstructing" a would-be "Stoic logic" in modern terms. The main trouble is in the fact that there is no general agreement, thus far, as to what is to "reconstruct", technically speaking.

[2]As it appears, the claim is neither new nor very original, but some recent authors have claimed otherwise, in the meantime. Rather unconvincingly, *on technical grounds alone*, to my mind. Cautiously, I shall, however, avoid, *prima facie*, polemic remarks on current research attempting to show something else. For convenience, the discussion of the *Stoic quantifiers* and a detailed scrutiny of the sources – in guise of supporting *textual evidence* for my remarks – are deferred and will appear as separate notes.

[3]Cf., e.g., Atherton 1993, on this.

[4]In proof theory we need not be concerned with their metaphysical status.

[5]Contradiction is at the root of Ancient (Greek) logic. Roughly, for the Ancient Greeks to prove something is to prove a contradiction. Worth mentioning here is the pre-Stoic tradition on this subject: the Pythagoreans, with their – somewhat empirical – tables of *opposita*, Heraclitus' metaphysics of conflict (*polemos*), the Eleats' obsessive interest in contradictory arguments and in *reductio ad absurdum*, the Sophist's specious, somewhat defective and argutious use of contradictions, Socrates' maieutic "art" of refuting an

opponent, *en tō agora*, Aristotle's "square of oppositions" [*Peri herm.*], his *very* specific concern with defective refutations [*Soph. el.*], etc.)

[6]No relation to Anderson & Belnap 1975, 1992, where the technical term "entailment" is reserved for specific conditional propositions, as expressed by formulas. The closest Anderson-Belnap approximation would be, likely, "first degree entailment", in this context.

[7]Incidentally, Frege had two "logics": a *Satzlogik*, presented axiomatically, in his BS, and a *Regellogik*, in *Grundgesetze der Arithmetik 1*, 1893 [GGA]. On the latter, see, however, Note 12 below and, possibly, Rezuş 2009.

[8]The special case $n = 1$ (pointing out to so-called "monolemmatic" arguments) is also attested in extant Stoic texts, although not explicitly so in Chrysippus. In particular, even if our sources are not very clear on this, entailments of the form $\mathrm{A} \vdash \mathrm{A}$ (expressing the "law of identity") were, curiously enough, not accounted for as (valid) "syllogisms". (Perhaps on the reason they do not necessarily involve logical connectors.) However, this is a mere terminological detail – after all, Aristotle did not call the monadic entailments "syllogisms" either, yet he recognised valid *immediate inferences* [later terminology] $\mathrm{A} \vdash \mathrm{B}$, for specific propositional "types" A, B –, since, given the Stoic way of understanding and explaining entailmens (via rejections / refutations / (logical) conflict), an entailment of the form $\mathrm{A} \vdash \mathrm{A}$ is to be accounted for as being equivalent with the "law of [non-] contradiction" stated in terms of polar oppositions. And it did not occur to Chryssipus to deny the validity of the latter or to defend a would-be "paraconsistent" logic, as in the case of some moderns. (See details below.) Similar remarks apply, *mutatis mutandis*, to entailments of the form $\mathrm{A}, \mathrm{B} \vdash \mathrm{A}$, resp. $\mathrm{A}, \mathrm{B} \vdash \mathrm{B}$ or, more generally, $\Gamma \vdash \mathrm{A}_i$ (where $0 < i < n + 1$, and Γ is as above).

[9]In the modern sense (i.e., valid entailments taken as primitive). The Stoics used *axiōma* as a technical term for "proposition".

[10]For technicalities, see, e.g., the PhD Dissertation of Katerina Ierodiakonou [*Analysis in Stoic Logic*, London 1990], and Susanne Bobzien's monograph on *Stoic syllogistic*, 1996. Cf. also Bobzien 1999, 2016.

[11]In this sense, the Stoic *distinguo* between entailments and conditionals (expressed here by implicative formulas) is close to contemporary (post-Fregean) conceptual standards, and, in a way, superior to Frege, who did not make such a distinction in his GGA. Actually, Frege won't have understood the point behind the so-called "deduction theorem" (Tarski 1921, Herbrand 1934). See also Rezuş 2009.

[12]Assuming a world without gradual transitions, shadows, dawn, twilight

zones and so on.

[13]We can normalise this situation – "syntactically", so to speak –, by fiat, taking "non" as a formal indicator for polar opposition in "variable" atoms, and write, e.g., non(A), for any "variable" atom A. Finally, the choice is arbitrary, of course, as we end up with (language) meaning postulates of the form A = non(non(A)), for every such an atom A. (Cf. with the use of literals in recent approaches to proof theory.) See, however, the remarks on "syntax" vs "semantics" in Stoic logic, following below.

[14]The Ancient's preferred examples would have rather been of the kind "the part is equal to the whole", for $\perp$, and "the part is less than the whole", for $\top$.

[15]Put things on a sphere – or on a circle, for that matter – to see the point behind the ad hoc "polar"-terminology. If I am living in Western Europe, my Canberran friend, Bob, is my "polar", from my point of view, and conversely, from his, so that each of us is the "polar of a polar" etc.

[16]This is much similar to the way some recent proof-theorists would present classical logic. Cf., e.g., Pyotr S. Novikov, Kurt Schütte, or, *mutatis mutandis*, Jean-Yves Girard, in his "linear" logic.

[17]"Multiary" links, like in the case of $\wedge$ (conjunction) and $\vee$ (inclusive disjunction) are to be taken, again, as a feature of the natural language and are "resolved" / analysed into binary links / connectors, in the obvious way.

[18]Incompatibility is exemplified in the third Stoic indemonstrable T3. Incidentally, Charles Sanders Peirce (1839–1914) re-discovered the Stoic **nand** before Henry Maurice Sheffer (1882–1964), but he did not manage to publish his finding.

[19]Given the method of construction (by "polars"), it is enough to attest a single member of each polar pair in our texts. Only case [4] does not occur explicitly in Chrysippus (it appears in later Stoic textbooks, however). The above correspond to the modern truth-functional (or Boolean) binary connectives (see below). In particular, in case [2] and [3], one must have, semantically, "conjunctions" $A \nrightarrow B = (A \wedge \mathrm{opp}(B))$, and $A \nleftarrow B = (\mathrm{opp}(A) \wedge B)$, resp., so that the Stoic intended readings *mãllon* A *ē* B, and *ē̃tton* A *ē* B, resp. would have been quite intuitive, in the end, granted the fact that the official Stoic negation could have been defined in terms of polar oppositions, as not(A) := opp(A), for any (complex) A. This terminology has nothing to do with would-be "comparative" (non-truth-functional) propositions, and the like, as some recent readers of Chrysippus used to speculate. See also below. — Apparently, however, Chrysippus and his followers thought of

"intensional" (non-truth-functional) connectors, as well. Our sources are not very illuminating on this subject, though.

[20]The latter can be viewed as "disjunctions" as well as "conjunctions", and can be analysed into / reduced to / defined in terms of proper connectors. Otherwise, our texts confirm the corresponding equivalences.

[21]By pondering upon Apollonius Dyscolus *Conj.* [Schneider GG II.i] 222.25–26. Here, round parantheses are mine – where "either" might have been left out, by modern standards –, i.e., one must have $((A \vartriangle B) \wedge A)$ resp. $((A \vartriangle B) \wedge B)$.

[22]A better name for the polar construction – implicit in the Stoic way of understanding logic – would be perhaps "proof-theoretic semantics" (as in recent work of Dag Prawitz and some of his followers), and we can take the method as making up *the right way of justifying classical logic, conceptually*.

[23]Cf., e.g., Schütte *et al.*

[24]A terminological aside: J.-Y. Girard used the terms "additive", "multiplicative" and "polarity" in a different sense.

[25]This is just a convenient shorthand, to save repetitions.

[26]This does not make, as yet, the underlying logic "classical", as one might be tempted to think at a first look. Indeed, the construction "by (classical) polarities", sketched in the above, would also apply to a couple of so-called "substructural" logics, as well (like the Anderson-Belnap relevance logic **R**, the so-called non-distributive **R**, or "Lattice **R**" [**LR**], and, even, Girard's "(classical) linear" logic **LL**, for instance [by the Anderson / Belnap standards, **LL** is "non-distributive", by the way]).

[27]With this notation, $\Gamma \Vdash$ means actually $\Gamma \vdash \perp$.

[28]The moderns would likely want to have a limit case, i.e., an additional "axiom" concerning the empty sequence (nil). On technical reasons, it is appropriate, indeed, to identify the empty sequence (nil) with the constant atom $\top$ [sic], having thus $\top \vdash A \Leftrightarrow \vdash A$ ("theoremhood", for A), as usually in our textbooks.

[29]Gentzen had *Schnitt*, "cut", instead of Frege's *(Ketten)schlusss*, whence also our current way of speaking in proof-theory. Apparently, Gentzen borrowed the idea from Paul Hertz (1881–1940), a former physics student of David Hilbert, not from Frege's GGA (I am trusting Paul Bernays on this, who actually supervised Gentzen's Göttingen Dissertation). As regards terminology, in English, dilution (dil) is oft referred to as "weakening", an inadvertent translation from German, where one has *Verdünnung* (like for wine and/or in chemistry).

[30]In modern logic, the restrictive proviso (C|Γ) can be properly formalized in *witness theory*, by using an explicit λ-calculus notation for "witnesses" (formal proofs), as, e.g., in N. G. Bruijn's AUTOMATH proof-checking systems, and, in general, in logic systems / calculi based on the so-called "Curry-Howard Correspondence [or Isomorphism]".

[31]Or else, in plain English: *contradictions are infectious*.

[32]This *contra* Jonathan Barnes [cf. his *Proof destroyed* (1980)], Susanne Bobzien (1996), Marek Nasieniewski (1998) *et al.*, who suggested that Chrysippus' logic should be rather viewed as "a kind of relevance logic" [sic]. With reference to the discussion of (dil) above, a "true relevantist" (i.e., a paradigmatic relevance logic defendor) would have likely talked about "relevantly [in-] consistent" sequences – or sets – of propositions, instead. A correct formal model-theoretical account of the alternative goes, however, far beyond the techniques Chrysippus and his followers would have had at hand. As the alternative reading is quite tempting, I shall examine in detail the Barnes-Bobzien suggestion in a sequel to these notes.

[33]A "logical principle", as, e.g., Ierodiakonou 1990, II.2.2 had it, while insisting on the absence of the *definite* article (before *theōrēma*) in the Sextan text (which I took seriously).

[34]Expressed by a 'CUT-formula', in Gentzen's terms.

[35]Quite unlikely, but why not?

[36]By Gerhard Gentzen, via Paul Hertz, namely.

[37]The reader has already realised, by now, that the polar statement of the proper logical rules (those involving connectors) above amounts to an *exact copy* of the Gentzen **LK**-rules [for classical logic], taken modulo ($\vdash$), and including inversions (i.e., as equivalences, instead of unidirectional meta-conditionals).

[38]Note that we have both tautology- as well as rule-completeness, thereby.

[39]Even if the polar construction suggested here would have no historical support in the Stoic texts, the *conceptual justification of classical logic* should rest on the very same principle, in the end, a rather simple fact that has been overseen – to my knowledge – by, virtually, all defendors of classical logic, so far. As I got the idea by paying attention to Chrysippus and his followers, in the first place, I'd better credit him with the finding: the present remarks make up just a piece of (historical) data-retrieval, indeed. In fact, the entire construction is based on *a very Greek idea* (in the ancient sense), there is nearly nothing to wonder about. Yet, *fallait-il y penser !*

[40]Whether this was actually the case in Chrysippus and his followers, we

cannot tell for sure: the supporting positive evidence in the extant texts is rather scarse.

[41]Just define $\Gamma \Vdash \Leftrightarrow_{df} (\Gamma \vdash \bot)$, in **Ch**[$\vdash$].

[42]In view of what has been said before, it is a simple exercise to "reconstruct semantically" the full **Ch** along the [$\wedge$-$\triangle$]-fibration, say. See, however, **§3** below.

[43]See, e.g., *In Anal.Pr.* 389,31–390,19 = Hülser **1137** and **1083**, resp.

[44]By Chrysippean standards, the Peripatetics used to deal only in atoms [atomic propositions], so to speak. — As an aside, in this guise, the generous idea of some of the *recentiores* to correct Łukasiewicz 1957 (who, otherwise, confused *use* and *mention*, in his heroic endeavours of "reconstructing" Aristotle "from the standpont of modern formal logic"), on the basis of a would-be genuine, alternative idea of "natural deduction" is, at best, a terminological *quiproquo*, since what we usually call *natural deduction* in contemporary logic – the Jaśkowski-Gentzen-Fitch-Prawitz-etc. approach – consists, *prima facie*, of an attempt to characterise, theoretically, the behaviour of the *logical connectives*, things unheard of in Aristotle and badly mishandled in the later Perpipatetic lore, as well as in the very learned ruminations of "historians" of logic à la Carl [von] Prantl, Eduard Zeller and the like.

[45]Cf. the (rather deceiving) comments of Jonathan Barnes on this, in Barnes 1985.

[46]Gentzen's **LK** is based on an ad hoc choice of primitives, reflecting, most likely, his casual interest in the Heyting logic, which, reputedly, is *an empirical construction*. (See the discussion of the first-order Heyting logic, viewed as a proof-theoretic fragment of classical logic – in witness-theoretical terms [$\lambda\gamma$-calculus] –, as appearing in Rezuş 1991, 1993.) It never occurred to any intuitionist logician – or mathematician – to *justify* (*conceptually*) the choice of the primitives in the "standard" propositional intuitionistic signature [$\neg,\rightarrow,\wedge,\vee$], or else in the "reduced" one [$\bot,\rightarrow,\wedge,\vee$]. Indeed, why not an intuitionistic **nand** [incompatibility], or an intuitionistic **nor**, for that matter? Or else – in view of the fact logic was (in the mind of L. E. J. Brouwer) *an empirical affair* –, should we, perhaps, expect discovering a hundred – or two – of *new intuitionistic connectives* (new atoms of logical meaning), during the first quarter of the current millenium, like, *mutatis mutandis*, in physics?

[47]Without further additions in the primitive syntax, they can be recovered only by an algebraic trick, like in groups, by proving, e.g., first that "all zeroes are equal" [unlike in groups, in a Boolean algebra we have two "zeroes", idest

$\top$ and $\perp$, not a single one], i.e. by something like: (1) define first $\top[C] :=$ $(C \rightarrow C)$ and (2) show next that the rules imply: $\top[A] = \top[B]$, for all A, B, in the sense of material equivalence **iff**, and analogously for $\perp[C]$.

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