Beyond BHK

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equivalences

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[...] *intuitionist mathematics has its general* [...] *theory of mathematical assertions*,
a theory which [...] *may be called* `intuitionist mathematical logic.`
L. E. J. Brouwer [*Brouwer Archief*, MS 49 (tr. W. P. van Stigt)]

Foreword

Categories of objects. Logics as equational proof-theories. Traditionally, the "language" of classical first-order logic has means of referring to *"individuals"* in some domain and means of expressing "facts" (or *propositions*) about them. The *proofs* themselves are handled in a *non-objectual* way. In particular, they are not codified syntactically. Some other kind of representation is implicitly involved in so-called "proof theory". Usually, the latter makes appeal to the *eye* – as opposed to the *mind*! –, rendering any *theoretical approach to logic* debatable (logic is *about proofs*, after all). This policy hides a rudimentary form of empiricism and is common to both pre- and post-Fregean traditions. Among other things, it forbids subjects like "proof-semantics for classical logic", for instance. On the same basis, it doesn't make too much sense to look for (theoretical) *criteria of proof-identity*, beyond obvious isomorphisms that can be established on "proof-figures", "proof-trees" or some other *ad hoc* organization of the visual representation space.

There is room for a different way of thinking about proofs, however. We are able to *recognize* and *identify* proofs on *theoretical grounds*.

Besides individual terms and formulas, we construct first-order languages with an additional syntactic category: the *proof-terms*. The intended meaning of a proof-formalism is given by the *propositions-as-types* isomorphism [H. B. Curry, C. A. Meredith, H. Läuchli, W. A. Howard, etc.]: the propositions are the "types" of proofs (or "proof-classifiers").

A *category of objects* is (analogous to) a constructive set (*à la* Bishop or Martin-Löf). In intuitionistic/constructive mathematics, the proofs ("derivations") make up a category of (abstract) objects: understanding proofs require

1° (generic) means to refer to *arbitrary objects* of the category (*proof-variables*, *proof-terms*) and

Dedicated to Dirk van Dalen, on His 60th Anniversary (March 13, 1993). On this occasion, a slightly different version of the *Foreword* has been printed as an "extended abstract" of the full paper in the **Dirk van Dalen Festschrift**, edited by Henk Barendregt, Marc Bezem, and Jan Willem Klop, as *Questiones Infinitae V* [Publications of the Department of Philosophy, Utrecht University], Utrecht 1993, pp. 114–120.

2° *identity-criteria* for these objects.

The proofs of a first-order logic L (containing the so-called "positive" implication) can be described *via* an equational theory $\lambda(L)$, viz. a typed/stratified λ-calculus, extending the ordinary typed λ-calculus λ^τ. So far, this procedure has been known to apply only to the first-order intuitionistic logic **HQ** and some derivatives, as obtained by adding, e.g., propositional quantifiers to the propositional fragment of **HQ** or higher-order classifiers. As applied to **HQ**, this is the essence of the so-called "**BHK**"-interpretation ["**B**rouwer-**H**eyting-**K**olmogorov"] of the intuitionistic proof-operations [Troelstra & van Dalen 88]. We can extend this technique "beyond **BHK**". E.g., [Rezuş 90] contains a treatement of classical first-order logic on these lines, while [Rezuş 91] is concerned with the interpretation of the equational theory of first-order proofs in modal logics (contained in Lewis' **S5**).

Which proofs are 'reliable' within the classical proof-world? In these notes, we examine the equational proof-theory of the first-order intuitionistic logic ["the Heyting calculus"] *as a subsystem of classical proofs*. In other words, we ask rather, in "positive" terms: *what is a 'reliable' proof* [*for Brouwer and Heyting*, Dutch: *betrouwbaar*], *within the classical proof-world?* Obviously, the extant (classical, non-intuitionistic, post-Gentzen) tradition in proof-theory (*Beweistheorie*) has no means to answer this type of question. [In fact, the question cannot be even formulated.]

For an intuitionist, a *proof* is an (abstract) *object* (of thought), occurring naturally in the current (mathematical) practice, as a result of a *systematic reflection* on this practice. As expected, intuitionistically, a *proof-theory* is a piece of (intuitionistic) mathematics.

[*Classical*] *reductio ad absurdum as an abstraction operator*. For the first-order classical logic **CQ** or its extensions, the required additions – to some background knowledge of typed λ-calculus – presuppose the fact that [1°] we are able to define, e.g., a proof-operator [an abstractor, say, i.e., a *proof*-variable binding mechanism], recording the genuinely Boolean uses of *reductio ad absurdum* and that [2°] we can describe its equational behavior. (There is a *combinatory* alternative to this, less transparent, however.) This operator is the γ-*abstraction* [Rezuş 90]: intuitively, it allows to conclude "positively" that a proposition expressed by a formula A has a proof [γx:¬A.e⟦x⟧, say] from the fact that a proof [e⟦x⟧] of a contradiction $\perp$ [= *falsum*, *absurdum*] has been obtained from the ["negative"] assumption that there is an arbitrary proof [x] of ¬A. (Everywhere here, negation ¬ is supposed to be understood inferentially [$\neg A \equiv_{df} A \to \perp$].)

The general concept disclosed is that of a [*typed*] $\lambda\gamma$-*theory* [= Post-consistent extension of λ^τ]. In particular, the proof-theory of the first-order classical logic **CQ** can be formulated as a typed λ-calculus λ(**CQ**).

We build upon a typed λ-calculus $\lambda\pi!$, which is familiar from the *Automath* literature and Martin-Löf's type-theory: besides the usual typed abstractions and applications, $\lambda\pi!$ has *type-products* [represented by conjunctions] with an *extensional pairing* ["surjective pairing"] and is augmented by first-order abstractors and applications associated to *first-order products of families of types* [represented by universal quantifications]. In fact, $\lambda\pi!$ is, more or less, the pure part of an *Automath*-system, proposed in [Zucker 77]. Proof-theoretically, it describes also the proof-behavior of the [$\perp,\to,\wedge,\forall$]-fragment of Johansson's [36] *Minimalkalkül*. $\lambda\pi!$ is known to be Post-consistent [*qua* equational theory].

$\lambda\gamma_{(0,\&)}$**CQ**: *the proof-theory of first-order classical logic*. The most economical formulation for a λ(**CQ**)-theory is likely a calculus $\lambda\gamma_0$**CQ**, which extends properly $\lambda\pi!$, by primitive γ-abstractions γx:¬A.e⟦x⟧, where A is *atomic* ("prime"), subjected to the obvious proof-term *stratification rule*:

($\to i\gamma$): Γ[x:¬A] ⊢ e⟦x⟧ : ⊥ ⇒ Γ ⊢ γx:¬A.e⟦x⟧ : A,

for any assumption-set Γ, and – besides γ-congruence – only two equational postulates, stating, resp.

[$\eta \to \gamma$]: γ-*extensionality*, i.e., unicity of the γ-behavior relative to the usual typed applications [= uses of *modus ponens*], [formally, assumming stratifiability on both sides, one has a reversal of the usual η-rule:

γx:¬A.x(f) = f, provided f does not depend on x], and

[$\oint\gamma$]: γ-*diagonalization*, allowing to eliminate specific uses of *reductio ad absurdum* occurring within the scope of ["inside"] a proof by *reductio* "of the same type", so to speak. [Assumming stratifiability on both sides, this reads, formally:

γx:¬A.f⟦x⟧(x(γy:¬A.e⟦x,y⟧)) = γz:¬A.f⟦z⟧(e⟦z,z⟧),

where e⟦z,z⟧ is obtained from e⟦x,y⟧ by identifying the displayed proof-variables [x ≡ y ≡ z]. Intuitively, a proof of the form γx:¬A.f⟦x⟧(x(γy:¬A.e⟦x,y⟧)) has, indeed, the character of a – slightly sophisticated – "proof-détour", since γ plays, classicaly, the rôle of an "introduction" rule, rather than that of an "elimination". However, the "*int*roduction-*elim*ination" dichotomy applies properly only to *Minimalkalkül*-like systems of rules/proof-operators.]

Note that A is supposed to be "prime", in both [$\eta \to \gamma$] and [$\oint\gamma$].[1]

The "complex" uses of *reductio* [γ-abstractions γx:¬A.e⟦x⟧, where A ≡ ⊥ or a complex formula] can be then introduced by an *inductive definition*.[2]The full theory $\lambda\gamma_{\&}$**CQ**, obtained by using arbitrary typed γ-terms as primitives, and by replacing the inductive conditions ([$\gamma\bot$], [$\gamma\to$], [$\gamma\wedge$], [$\gamma\forall$]) by obvious postulates is (stratification/equationally) equivalent to $\lambda\gamma_0$**CQ**.

We can show **Cons**($\lambda\gamma_0$**CQ**), i.e., Post-consistency for $\lambda\gamma_0$**CQ** [= proof consistency for first-order classical logic], by extending the kernel of the "negative" translation of [Glivenko 28,29] to the proof(-term)s of $\lambda\gamma_0$**CQ**. The outcome [**III**, below] is an *effective* (1–1) *translation* from $\lambda\gamma_0$**CQ** to its γ-free fragment $\lambda\pi!$, known to be Post-consistent. In [Rezuş 90], the result is obtained by a *type-free* argument. In particular, the procedure – described on half a page – is also *admissible* intuitionistically: it supplies a "dictionary" for proof-operations that do not make sense, in general, in the intuitionist practice.

What is *not intelligible intuitionistically* is just the *intuitive identification/interpretation of the abstract γ-operations as logical proof-operations*: indeed, the abstraction operator corresponding, classically, to *reductio ad absurdum* has only *local* – "finitary", so to speak – *meaning* in terms of **HQ**-proofs. Technically, for each [⊥,→,∧,∨,∀,∃]-formula A and any proof f such that f proves intuitionistically ¬A ∨ A, there is an abstraction operation γ_f, depending on f, which can be shown to share relevant properties with the "global" γ-abstractor. Beyond the – intuitionistically – recognizable "local"/"finitary" information, the classical γ-abstractor attempts to supply "global" information about proofs, acting, *qua* information-processing agent, like a *highly predictable* – although *intuitionistically "unreliable"* – *oracle*.

$\lambda\gamma_{\&}$**CQ** has [∨,∃]-"type-constructors" defined *via* the usual Ockham/De Morgan transformations [A∨B $\equiv_{df}$ ¬(¬A∧¬B), with ∃ "generalizing" ∨, as expected]. These definitions admit of associated "negative" proof-operators [Boolean "injections" **j**, **J** and Boolean "selectors "$\bigvee_{\natural}$, $\bigvee_{\cup}$], with an appropriate *extensional* behavior.

The [*full*] *Heyting calculus is a proper fragment of* $\lambda\gamma_{\&}$**CQ**. The bulk of the work [**IV**–**VII**] is devoted to the tedious task of deriving the stratification/equational behavior of the *Minimalkalkül*, resp. Heyting proof-operators associated to [⊥,∨,∃] in terms of the "negative" Boolean analogues. The full Heyting first-order proof-calculus λ**HQ**, with "⊥-conversions" [*ex-falso*-rules, here ω-*rules*] and so-called "commuting

[1]For readers familiar with the proof-system of the Heyting first-order logic, the γ-diagonalization postulate must be also reminiscent of the Heyting "commuting conversions". Indeed, the "selectors" associated to the intuitionistic ∨ and ∃ are analogous to appropriate uses of γ. In general, the Heyting proof-calculus would allow only "cancelling" γ-abstractions, of the form ω_A(e) $\equiv_{df}$ γx:¬A.e, where e : ⊥ does *not* "depend on" the assumption [x:¬A], so that this structural analogy is useful only in a Boolean setting.

[2]The "recursive eliminability" of γ was, in fact, known to/implicit in [Prawitz 65], although not in type-theoretic – or $\lambda\gamma$-calculus – form. This does not depend on γ-diagonalization, by the way; see details in [Rezuş 90].

conversions" ([Prawitz 65], [Troelstra 73], [Troelstra & van Dalen 88]) is shown, finally, to admit of a definitional embedding into the classical theory $\lambda\gamma_{\&}\mathbf{CQ}$.

The simulation of the intuitionistic proof-operations in terms of $\lambda\gamma_{\&}\mathbf{CQ}$ proof-operations is an *effective translation*. This yields $\mathbf{Cons}(\lambda\mathbf{HQ})$, too. Ultimately, we obtain an effective translation of $\lambda\mathbf{HQ}$ into $\lambda\pi!$, i.e., into a *proper fragment* of it, known to be consistent by intuitionistically/constructively acceptable means (cf., e.g., [Martin-Löf 84]). In fact, $\lambda\pi!$ and λ^{τ} can be shown to be *equi-consistent* by very simple translations, so that $\mathbf{Cons}(\lambda\mathbf{HQ})$ amounts, in the end, to $\mathbf{Cons}(\lambda^{\tau})$, i.e., the problem is reduced to a question about the ordinary – "simple" – typed λ-calculus.[3]Existing alternatives of showing $\mathbf{Cons}(\lambda\mathbf{HQ})$ recommend either a *confluence* argument or a *model-theoretic* approach. The former involves hundreds of separate cases to check, whereas the latter has meager chances of becoming intuitionistically intelligible.

The logic of "complete refutability". In analogy with $\lambda\mathbf{HQ}$, one can isolate – within $\lambda\gamma_{\&}\mathbf{CQ}$ – the equational proof-theory of the first-order logic of "complete refutability" **DQ** (also known as a "logic of strict negation": Curry's *LD* in [Curry 52,63], [Seldin 89]). The interest in **DQ** is in the fact that its inferential [⊥,→] part is, in a sense, "non-Brouwerian" and – inside **CQ** – *complementary* to the inferential part of **HQ**. Indeed, **DQ** and **HQ** disagree mainly on negation: **DQ** allows "the Law of Clavius" [*consequentia mirabilis*: $\neg A \to A \to A$], which is, clearly, not **HQ**-derivable, and discards as incorrect ("non-strict") the intuitionistically unobjectionable *ex falso quodlibet* [$\bot \to A$].

On a proof-level, behind the "Rule of Clavius" $\neg A \Vdash A \Rightarrow \Vdash A$, there is an inferential proof-operation [i.e., an abstraction operator] ε, say, which is definable classically, in $\lambda\gamma_{\&}\mathbf{CQ}$, by $\varepsilon x{:}\neg A.a[\![x]\!] \equiv_{df} \gamma x{:}\neg A.x(a[\![x]\!])$. So, the type-theoretic variant of this – strange, old – proof-pattern[4]is:

$$\Gamma[x{:}\neg A] \Vdash a[\![x]\!] : A \Rightarrow \Gamma \Vdash \varepsilon x{:}\neg A.a[\![x]\!] : A.$$

Now, the ω's mentioned above [that is: the uses of *ex falso quodlibet*: $\omega_A(e) \equiv \gamma x{:}\neg A.e$, with x not free in e], and the "Clavian" abstractions ε are "complementary" within/inside $\lambda\gamma_{\&}\mathbf{CQ}$, in the sense that they can be put together, in order to make up [the effect of] a use of *reductio ad absurdum* [γ]. On a provability level, this is well-known. Formally, one can define, in $\lambda\gamma_{\&}\mathbf{CQ}$,

$$\gamma^{\circ}x{:}\neg A.e[\![x]\!] \equiv_{df} \varepsilon x{:}\neg A.\omega_A(e[\![x]\!]),$$

on a proof-(term)-level, and, in view of (a special case of) [$\oint\gamma$], $\lambda\gamma_{\&}\mathbf{CQ}$ is able to identify the new γ°-abstraction with the old one. This can be generalized to a hierarchy of triples [$\gamma^{[n]}$,$\varepsilon^{[n]}$,$\omega^{[n]}$], $n \geq 0$, that are collapsed back to [γ,ε,ω] by γ-diagonalization. Without [$\oint\gamma$] or with assumptions weaker than [$\oint\gamma$], one can distinguish a hierarchy of subsystems of $\lambda\gamma_{\&}\mathbf{CQ}$, that "prove the same theorems" [i.e., share the same "stratification criteria" for proof-terms], but are still equationally distinct (using different "identity-criteria").

The addition of the ε-abstraction to the ordinary typed λ-calculus λ^{τ} yields the proof-theory of the inferential [⊥,→] part of **DQ**. The "positive" [→,∧,∀] part of **DQ** is like in **CQ** and in *Minimalkalkül*, whereas the "negative" [∨,∃]-proof-operators of **DQ** are slight generalizations of their *Minimalkalkül*/ intuitionistic analogues. Ultimately, one can reconstruct proof-theoretically **DQ** as a typed λ-calculus, $\lambda\mathbf{DQ}$ say, that can be embedded into $\lambda\gamma_{\&}\mathbf{CQ}$, as in the case of the Heyting proof-calculus. A complete description of $\lambda\mathbf{DQ}$ is tedious (although it has no "⊥-conversions" - since it has no ω's - it requires more "commuting conversions" than **HQ**): we give only information estimated useful in view of retrieving the basic ingredients (*modulo* patience and computing power).

A conjecture. Internal evidence (as, e.g., among other things, the fact that $\lambda\gamma_{\&}\mathbf{CQ}$ is complete relative to the Heyting calculus, that it collapses many distinctions, like those among [γ,ε,ω]-triples, etc.) suggests the *conjecture* that $\lambda\gamma_{\&}\mathbf{CQ}$ is *Post-complete*. That is: if $\mathbf{CQ} \Vdash A$ one cannot add a new closed equation

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[3]The Heyting calculus is *not* contained in Martin-Löf's type theory (because of the "⊥-conversions" and the "commuting conversions").

[4]Cf. Euclid *Elementa* IX.12, [Clavius 1611] *ad loc.*, [Cardano 1663] 4, 579, [Saccheri 1697,1733] *passim*.

$a_1 = a_2$, where a_1, a_2 are proofs of A, without loosing [Post-] consistency for the resulting extension. This is a typed analogue of a situation obtaining [Böhm 68] for the *extensional type-free λ-calculus*, where we cannot identify consistently two arbitrary normal forms. Here, the normal[izabi]lity requirement is already insured by stratifiability.

Coda. The *appendix* pays attention to *assorted topics*, somewhat loosely related to the main theme (the Gentzen *L*-systems, the concept of a proof-transformation for **CQ**, and the "double negation" interpretations of **CQ** into **HQ** along the Kolmogorov [25] translation-pattern). Last, a systematic *guide* to the *essential* literature on **BHK** is supplied.[5]

[5]Wherefrom one can see in a glance that the main theme is surprisingly endemic in print and has many ramifications. The *guide* is also intended to cover the [remaining] "assorted topics", alluded to – but not documented – in the main text, as a concise – and, hopefully, more useful – substitute for would-be learned – yet, inevitably verbose and ultimately casual – footnotes. There is no claim of completeness, as regards the corresponding *bibliography*, of course.

Chapter I

Background: the global structure of a proof-language

General assumptions. A (first-order) *proof language* is an extension of a usual first-order language (with, as syntactic categories, the *individual terms* and the *formulas*). Typically, we take also into account a new syntactic category: the *proof-terms*.

As regards the *intended meaning* of the resulting formalisms, we use the familiar [Curry-Meredith-Howard] *propositions-as-types* isomorphism as a way of speaking: a *proposition is* therefore nothing else than *the type of a proof*-[*object*], denoted by a *proof-term*. Indeed, a (first-order) proof is the proof of a *fact*, a *proposition*, a [Fregean] "*thought*", etc. that can be expressed by a *first-order formula*. In other words, we assume implicitly that the propositions of a given logic L can function as *regimentation criteria* ("classifiers" or "types") for a special *category of objects* that can be *identified intuitively* with the *L-proofs* (= the *proof-objects of L*). Here, a *category of objects* is analogous to an intuitionistic *species* or a *constructive set*, in the sense of Errett Bishop and Per Martin-Löf. In short, we know *what is a category* **C** *of objects* if (1) we have means of establishing *what is an arbitrary object of* **C** and, (2) for any two arbitrary objects of **C**, we have also means of establishing *whether they are equal or distinct objects of* **C**. Of course, the exact nature of these "means" must be further qualified, for each **C** of concern.[6]

The basic assumptions in what follows are that, *for any first-order logic L*, *the L-proofs make up a category of objects* and that, moreover, this category *can be* also *described completely* within an appropriate *typed* λ-*calculus* $\lambda(L)$. The general concept to be disclosed is that of a $\lambda\gamma$-*theory* (or $\lambda\gamma$-*calculus*).

E.g., if L is the first-order classical logic **CQ**, $\lambda(L)$ is a proper (conservative) extension of the ordinary typed λ-calculus λ^{τ}, i.e., a *type*(d) $\lambda\gamma$-*theory* (in λ-calculus, a typed λ-theory is a Post-consistent extension of λ^{τ}). Here, the reference to γ points out to a specific [typed] *abstraction operator* [intended intuitive interpretation in proof-theory: the genuinely classical *reductio ad absurdum*].[7]On the other hand, however, a (typed) $\lambda\gamma$-calculus is, essentially, an equational system meant to establish the behavior of *abstract objects* and *admits of a* (*very specific*) *proof-interpretation* (only) as one of its possible *intuitive interpretations*. The identification (for a given logic L):

proof in L = meaning of a proof-term in some λ(L)

correlates thus an *intuitive notion* and a *technical concept*. Still, from a λ-calculus viewpoint, this identification is *theoretically dispensable* [it is a *claim about the world* and *not a piece of mathematical evidence*, so to speak]. *The most general interpretation* of the stratified $\lambda(\gamma)$-terms [*directed asynchronous process of information transfer under non-local control*, say] falls likely out of the scope of "proof-theory", in the traditional (post-Hilbert) sense.

Logics and type-theoretic presentations. Let **U** be a fixed *universe of discourse*. Here, **U** contains the *individuals* or the *referential points* of any (first-order) logic L. *A type-theoretic presentation of* L is a structure $[L] = <\Vdash_L, \tau_L>$, where $\Vdash_L$, τ_L are families of recursive predicates, such that

[6] Here, "distinct" means "falling apart", in Brouwer's terms. Notably, *sameness* and *apartness* were *primarily given* for Brouwer and could be "reduced" to each other only in special circumstances, while the "falling apart" (of objects of mind) could be grasped by *positive acts*, too. Whence a Brouwerian would eventually discard as inadequate the Western philosophical *topos* on *difference* (*Sophistes* 254B *sq.*) and the traditional (essentially Aristotelian) view on *negation* as a "primitive" (operation of the mind).

[7] Traditionally, referred to also under the label "*reductio ad absurdum*" is a specific instance of *modus ponens* (i.e., an intuitionistically correct rule) which allows to conclude that ¬A ($\equiv_{df}$ [A → ⊥]) has a proof from the fact that there is a proof of ⊥ (*falsum/absurdum*) under the assumption that there is an arbitrary proof of A. The *classical reductio* uses "negative" information, concluding "positively" to the existence of a proof of A from the fact that a proof of ⊥ can be obtained from an arbitrary *negative* assumption ¬A.

(1) $\Vdash_L = < \Vdash_L(\mathbf{U}), \Vdash_L(\mathbf{H}), \Vdash_L(\Lambda) >$ is *the family of syntactic categories of* L, viz.,
- $\Vdash_L(\mathbf{U})$ contains *the* **U**-*terms of* L,
- $\Vdash_L(\mathbf{H})$ contains *the* L-*formulas*,
- $\Vdash_L(\Lambda)$ contains *the proof-terms of* L,

(2) $\tau_L = < \vdash_L, =_L >$ generates the *category of* L-*proofs*.

In detail, $\Vdash_L(\mathbf{U})$ establishes the admissible ways of referring to **U**-*objects* (*individuals* in **U**); notation: $\Vdash_L \mathbf{t} :: \mathbf{U}$ (read: "**t** is a **U**-term (of/in L)"), $\Vdash_L(\mathbf{H})$ establishes the admissible ways of referring to *propositions* in L, notation: $\Vdash_L \mathrm{A} :: \mathbf{H}$ (read: "A is an L-formula"), whereas $\Vdash_L(\Lambda)$ establishes the admissible ways of referring to L-*proofs*, notation: $\Vdash_L \mathrm{a} :: \Lambda$ (read: "a is a proof-term of L"). Moreover, $\vdash_L$ is a *stratification criterion* for the proof-terms of L and $=_L$ is an *identity criterion* for L-proofs. So, for any (first-order) logic L admitting of a type-theoretic presentation $[L]$, the criteria specified by τ_L yield a *category of objects*. In other words, $\vdash_L$ establishes *the form of* L-*proofs*, answering the question *what is an arbitrary object within the category of* L-proofs, whereas $=_L$ allows to establish *whether two arbitrary objects falling within the category of* L-*proofs are to be accounted for as equal or distinct*. In traditional terms, $\vdash_L$ characterizes (in fact, elliptically so) the *consequence relation of* L. However, the post-Fregean (or, better, post-Gentzen) tradition in *classical* logic ignores any *proof-identity* criteria. The latter aspects have been developed theoretically, so far, only within the (post-Brouwerian) tradition of intuitionistic proof-theory, specifically within the so-called **BHK** [**B**rouwer-**H**eyting-**K**olmogorov] interpretation of the Heyting (first-order) logic.

The present point of view is meant to extend the kernel of the "**BHK**-interpretation" to the classical (first-order) logic (and, in fact, to any other "logic" which is worth being called so).

> One of the immediate implications of the assumptions above is that *the meta-theory of* (any) *logic* (= abstract system which owns the concept of a proof) *must be* also (a piece of) *intuitionistic* (mathematics). *Modulo* the choice of terms, this tenet constitutes, certainly, one of the (few) points of agreement between Brouwer and the formalist tradition.

The L-*atoms*. For a given first-order logic L, each syntactic category of $[L]$ is generated from *atoms* (*atomic primitives*). So the categories $\Vdash_L(\mathbf{U})$ and $\Vdash_L(\Lambda)$ are generated from *free atoms* or *indeterminates* (if L is a quantifier-free logic, $\Vdash_L(\mathbf{H})$ has free atoms, too).

Explicitly, for every (first-order) logic L, where $\mathtt{P}$ is an arbitrary primitive predicate symbol of L and $\mathbf{t} \equiv (\mathbf{t}_1,\ldots,\mathbf{t}_n)$, with $\Vdash_L \mathbf{t}_1,\ldots,\mathbf{t}_n :: \mathbf{U}$, the atoms of $\Vdash_L$ are:

- **U**-*atoms* (**U**-*variables*, *individual variables*) in $\Vdash_L(\mathbf{U})$: u, v,..., also decorated, in a set Var_u,
- **H**-*atoms* in $\Vdash_L(\mathbf{H})$, namely
  - $\mathtt{P}[\mathbf{t}]$, the "primes" of L, and
  - $\bot_L$ (*falsum*), standing for *a constant false proposition*, and, possibly, a primitive "dual" of $\bot_L$, viz.,
  - $\top_L$ (*verum*), standing for *a constant true proposition*,
- Λ-*atoms* (or *proof-variables*) in $\Vdash_L(\Lambda)$: x, y, z, ..., possibly decorated, making up a set Var_λ.

Within an actual type-theoretic presentation $[L]$ of L, the Λ-atoms of L are supposed to be *primitively regimented/"stratified"* [this regimentation depends only on the specific structure of $\Vdash_L(\mathbf{U})$ and $\Vdash_L(\mathbf{H})$]. Syntactically, the Λ-atoms are represented by "typed variables", where the "type-symbols" are all and only the objects generated by $\Vdash_L(\mathbf{H})$.

> In quantifier-free ("propositional") logics L, the "free atoms" of $\Vdash_L(\mathbf{H})$ are *propositional* (**H**-) variables, ranged over by p, q, r, ..., possibly with decorations and make up a set Var_h. In the general first-order case, there is no need for "free **H**-atoms".

For first-order logics we have a supply of primitive predicate symbols (constants) $\mathtt{P}$. The proper atomic types (the "primes") are constructed from these symbols and **U**-terms in the expected way. In particular, one may oft want to have a primitive *equality* predicate (denoted by δ_L, say) in a first-order setting.

Notation (*Global syntactic conventions*).

Where $\Vdash_L$ $\mathbf{t}$ :: $\mathbf{U}$, $\Vdash_L$ A :: $\mathbf{H}$, $\Vdash_L$ a,b :: Λ and u $\in$ Var_u, p $\in$ Var_h, x $\in$ Var_λ, the syntactic notations $\mathbf{t}[\![u]\!]$, $A[\![u]\!]$, $A[\![p]\!]$, $a[\![u]\!]$, $a[\![x]\!]$, $a[\![u,x]\!]$, etc. indicate the fact that the variables u, p, x occur with a "free" *régime* within $\mathbf{t}$, A, a, resp. where "free" and "bound" is to be determined in each case separately, relative to the specific structure of L. As ever, uniform substitution for the free $\mathbf{U}$- and $\mathbf{H}$-atoms is shown by $\alpha[\![u{:=}\mathbf{t}]\!]$ (for $\Vdash_L \alpha :: \mathbf{U}$ or $\Vdash_L \alpha :: \Lambda$) and $\alpha[\![p{:=}A]\!]$ (for $\Vdash_L \alpha :: \mathbf{U}$ or $\Vdash_L \alpha :: \Lambda$), resp. The uniform substitution operator for the free Λ-atoms of L (notation: $b[\![x{:=}a]\!]$) has a similar meaning, but is subjected to additional "type-constraints", depending on $\vdash_L$ (to the effect that x and a must be "of the same type").

$\mathbf{U}$-*terms*. In first-order logics, the $\mathbf{U}$-*terms* are constructed from $\mathbf{U}$-variables and a fixed stock of primitive function symbols f, in the usual way. One has, in fact, the *generic* inductive scheme:

($\mathbf{U}_0$) $\Vdash_L$ u :: $\mathbf{U}$, if u $\in$ Var_u,
($\mathbf{U}_\mathrm{f}$) $\Vdash_L$ $\mathbf{t}_1,\ldots,\mathbf{t}_n$ :: $\mathbf{U}$ $\Rightarrow$ $\Vdash_L$ $\mathtt{f}(\mathbf{t}_1,\ldots,\mathbf{t}_n)$ :: $\mathbf{U}$,

for any primitive n-ary f. For n = 0, f is a constant (standing for an element of $\mathbf{U}$).

Type-syntax. The syntactic category $\Vdash_L(\mathbf{H})$ yields the *type-structure* of $[L]$. As expected, $\Vdash_L(\mathbf{H})$ is constructed inductively from $\mathbf{H}$-atoms by closing under appropriate *type-constructors*. Traditionally, the latter correspond to *logic syncategoremata*; among them are the so-called "logical constants" and the quantifiers ("first-order" or "$\mathbf{U}$-quantifiers").

For propositional logics (with $\mathbf{H}$-atoms), we may also consider "propositional" or "$\mathbf{H}$-quantifiers". In the limit (Boolean) case, one obtains the so-called *extended propositional logic*.

In particular, any L admitting of a type-theoretic presentation is supposed to have a primitive binary type-constructor $\rightarrow_L$, standing for *implication in L*. In this setting, an *L-negation* is also available "inferentially", by $\neg_L A := [A \rightarrow_L \bot_L]$. Analogously, any first-order logic L which can be presented type-theoretically has at least a (first-order) *universal* $\mathbf{U}$-*quantifier* $\forall_L$.

Roughly speaking, this specific choice of type-primitives corresponds to the view that "logic is [at least] about inference and generality". In fact, this is also the *least* choice we *have to* make in order to have access to proper type-theoretic presentations. On a pure ideological level, one can also elaborate on the above, by saying that *there is no Toleranzprinzip in proof-theory* (with "proof-theory" taken in the *technical* sense of this paper). Indeed, an inadvertent choice of type-primitives might turn out to be fatal (theoretically prohibitive, etc.) at a later stage.

For any logic L admitting of a type-theoretic presentation $[L]$, the type-structure of $[L]$ can be described *schematically*. In detail, for any primitive n-adic atomic predicate P in L, where $\triangle_L :: [\mathbf{H} \longrightarrow \mathbf{H}]$ is a singulary (one-place) primitive type-constructor of L, $\circ_L :: [\mathbf{H} \times \mathbf{H} \longrightarrow \mathbf{H}]$ is a binary primitive type-constructor of L, and, for $X \in \{\, \mathbf{U},\mathbf{H} \,\}$, $\mathbf{Q}_{L[X]} :: [[X \longrightarrow \mathbf{H}] \longrightarrow \mathbf{H}]$ is the abstraction-operator corresponding to a primitive (first-order, resp. propositional) quantifier of L, the following inductive scheme gives the *generic notation* for the type-structure of $[L]$:

($\mathbf{H}$P) $\Vdash_L$ $\mathbf{t}_1,\ldots,\mathbf{t}_n$ :: $\mathbf{U}$ $\Rightarrow$ $\Vdash_L$ $\mathtt{P}(\mathbf{t}_1,\ldots,\mathbf{t}_n)$ :: $\mathbf{H}$,
($\mathbf{H}\bot$) $\Vdash_L$ $\bot_L$ :: $\mathbf{H}$,
($\mathbf{H}\top$) $\Vdash_L$ $\top_L$ :: $\mathbf{H}$, (if $\top_L$ is a primitive of L),
($\mathbf{H}\triangle$) $\Vdash_L$ A :: $\mathbf{H}$ $\Rightarrow$ $\Vdash_L$ $(\triangle_L A)$ :: $\mathbf{H}$,
($\mathbf{H}\circ$) $\Vdash_L$ A,B :: $\mathbf{H}$ $\Rightarrow$ $\Vdash_L$ $(A \circ_L B)$ :: $\mathbf{H}$,
($\mathbf{H}_X\mathbf{Q}$) $\Vdash_L$ $A[\![\xi]\!]$ :: $\mathbf{H}$ ($\xi \in$ Var) $\Rightarrow$ $\Vdash_L$ $(\mathbf{Q}_{L[X]}\xi.A[\![\xi]\!])$ :: $\mathbf{H}$, $[X \in \{\, \mathbf{U}, \mathbf{H} \,\}$, Var := Var_u, $\mathrm{Var}_h]$.

Given the above, the type-structure of any logic $[L]$ is fully specified by listing the primitive type-constructors (*syncategoremata*) of L.

Examples. Where → stands for *implication*, ∧ and ∨ for *conjunction* and *disjunction*, resp., and ∀, ∃ for the *first-order* quantifiers, [⊥,⊤,→,∧,∨,∀,∃] is a (very redundant) type-structure for the first-order classical logic **CQ**, the Heyting logic **HQ**, Johansson's *Minimalkalkül* **MQ**, etc. Of course, [⊥,→,∧,∀] and even [⊥,→,∀] should suffice for **CQ**.

> This scheme covers also type-structures for logics with *modalities*, *via* (**H**△), and/or *propositional quantifiers* [Π,Σ], *via* ($\mathbf{H}_X\mathbf{Q}$), for $X \equiv \mathbf{H}$ and Var := Var_h. One can obtain, e.g., a type-theoretic version of the (*classical*: Russell, Łukasiewicz, Tarski, etc.) "extended propositional logic" within a [→,Π]-type-structure: as expected, the resulting proof-calculi are equational extensions of the so-called "second-order typed λ-calculus". For a discussion of the proof-theory of (first-order) modal logics contained in Lewis' (first-order) **S5**, see [Rezuş 91].

First-order proof-contexts. If the atoms of [*L*] are specified, we can say what is a (*first-order*) *proof-context* (alternatively: an *"assumption set"*) for *L*, without making reference to the specific (type- and/or proof-operator-) structure of *L*.

An *assumption for L* is an *arbitrary* ("*virtual*") *proof*; notation: [x : A], where ⊩ x :: Λ (i.e., x is a free Λ-atom) and ⊩ A :: **H**. So, the assumptions are represented by typed/stratified proof-variables, where the "types" are denoted by the formulas of *L*.

For first-order logics, the assumptions are **U**-*parametric objects*: where ⊩ A :: **H**, the formula A may depend on a finite number of **U**-parameters $u_1,\ldots,u_m$, viz., on those **U**-variables that occur *actually* free in A. Formally, a **U**-*parameter* has the status of a *generic assumption* [u : **U**], where ⊩ u ::**U** ($u \in \text{Var}_u$). Still, **U** does not stand for a "type".

A (*first-order*) *proof-context* (*for L*) is a *finite sequence* Γ of assumptions and **U**-parameters for *L*, such that if [x : A] is an element of Γ and u (⊩ u :: **U**) occurrs actually free in A, then the **U**-parameter [u : **U**] is also an element of Γ. So, an arbitrary proof-context for *L* can be displayed as a *sequence* (better: a pair of sequences)

$\Gamma = [u_1 : \mathbf{U}]\ldots[u_m : \mathbf{U}] \smile [x_1 : A_1]\ldots[x_n : A_n]$, or
$\Gamma = [u_1]\ldots[u_m] \smile [x_1 : A_1]\ldots[x_n : A_n]$, or even
$\Gamma = [u_1]\ldots[u_m][x_1 : A_1]\ldots[x_n : A_n]$,

where the A_i's ($1 \le i \le n$) are formulas of *L*. Here, the separator "⌣" is a notational expedient (that can be absent) and is meant to contrast the **U**-*parametric component* (a "**U**-context") Γ_u and the *structural component* Γ_λ of $\Gamma = \Gamma_u \smile \Gamma_\lambda$. By convention, the empty sequence is a proof-context (notation: [ ]). If Γ_λ is the empty sequence, the **U**-parametric component Γ_u of $\Gamma = \Gamma_u \smile \Gamma_\lambda$ can be discarded (e.g., set Γ ≡ [ ], if no confusion can arise).

For logics with a *monoidal* [*simple, non-composite*] *consequence relation*, the *sequential* structure of the proof-contexts is, in fact, immaterial: one can use *assumption sets* instead of *sequences*, i.e., *sets* of assumptions and **U**-parameters. This simplification makes some "structural" rules redundant, although it is not suited for "hybrids" (Nuel D. Belnap Jr.), i.e., for logics with a *composite consequence relation*.

> *Propositional proof-contexts*. In the case of (propositional) logics with explicit *propositional quantifiers* the proof-contexts are **H**-*parametric* objects. That is, a *propositional proof-context* must be of the form $\Gamma = \Gamma_h \smile \Gamma_\lambda$, where $\Gamma_h := [p_1 : \mathbf{H}]\ldots[p_m : \mathbf{H}]$ is a sequence/set of **H**-atoms and, as earlier, $\Gamma_\lambda := [x_1 : A_1]\ldots[x_n : A_n]$, such that the A_i's ($1 \le i \le n$) have only **H**-parameters from among those occurring in the list/set Γ_h.

Proof-statements. For any logic *L*, the basic objects of [*L*] are the *proof-statements of L*. Specifically, [*L*] generates two distinct categories of proof-statements: the *inference statements* (in type-theoretic terms, we use to speak about *stratification conditions*) and the *proof-equations of L*.

An *inference statement* (a "*type-assignment*", a "*typing*", etc.) *for L* is of the form $\Gamma \vdash_L$ a : A, where Γ is a proof-context, $\Vdash_L$ a :: Λ and $\Vdash_L$ A :: **H**. As to its intended meaning, an inference statement is *an assumption form*; in other words, "$\Gamma \vdash_L$ a : A" is a statement about *a proof under assumptions* (reading "a proves A under the assumptions contained in Γ").

Intuitively, an inference statement $\Gamma \vdash_L$ a : A says that the assumptions of Γ, i.e., the "proofs" [x_i:A_i] contained in Γ, are taken "in *virtual* (proof-) space", whereas "a : A" is "an *actual* proof". So "proving" A in *L*, i.e., generating an inference statement of the form "$\Gamma \vdash_L$ a : A", can be viewed as an operation of *constructing an actual object* a "of type A" *from virtual information* concerning the "types" of the "ground components" of a. In fact, beyond the "virtual" information to be extracted from assumptions, this involves only a set of *L-specific* (*proof*)-*operators*, (given by) the so-called (proper) "derivation rules" of *L*.

A different kind of assumption form occurring in the proof-theory of first-order logics is epi-theoretic in nature and concerns the **U**-*parametrization* of the elements of $\Vdash$(**H**) and $\Vdash$(**U**). These are statements $\Gamma_u \Vdash_L$ A :: **H** or $\Gamma_u \Vdash_L$ **t** :: **U** and are meant to indicate the fact that (the formula) A, resp. (the **U**-term) **t** may contain free **U**-variables that are among those contained in the "**U**-context" $\Gamma_u \equiv$ [u_1:**U**]... [u_m:**U**]. Equivalently, we can write $\Gamma \Vdash_L$ A :: **H** and $\Gamma \Vdash_L$ **t** :: **U**, resp. instead, with the proviso that $\Gamma = \Gamma_u \smile \Gamma_\lambda$, for some assumption-list/set Γ_λ.

Notation (*Proof-contexts*).

(1) Unless otherwise stated, the notation Γ[x:A] $\vdash_L$ c : C means that the assumption [x : A] is not an element of Γ.
(2) The notation $\Gamma[\![u]\!]$ indicates the fact that the proof-context Γ contains the **U**-parameter u. If Γ = [u_1:**U**]... [u_m:**U**] $\smile\Gamma_\lambda$, Γ[u:**U**] or Γ[u] stands for $\Gamma_1[\![u]\!]$ = [u_1:**U**]... [u_m:**U**][u:**U**] $\smile \Gamma_\lambda$, where u is *not* free in the formulas occurring in Γ_λ and, moreover, [u:**U**] is not one of the [u_i:**U**]'s, $1 \le i \le m$.
(3) Where Γ_i, $1 \le i \le n$, are proof-contexts, their union is obtained by concatenating separately the corresponding Γ_u- and Γ_λ-sequences and is denoted by $\Gamma = \Gamma_1 \ldots \Gamma_n$. Of course, if the proof-contexts are thought of as being (assumption-) sets, the union of the Γ_i's, $1 \le i \le n$, is just *set*-union.
(4) We write $\Gamma \vdash_L$ a iff $\Gamma \vdash_L$ a : A, for *some L*-formula A, with $\Vdash_L$ A :: **H** (omitting *L*-subscripts, if no confusion can arise).
(5) Finally, $\Gamma \vdash_L a_1, \ldots, a_n$ is shorthand for the conjunction ($\Gamma \vdash_L a_1$) & ... & ($\Gamma \vdash_L a_n$).
(6) *Mutatis mutandis*, if *L* is a (propositional) logic with **H**-atoms and propositional quantifiers, one has analogous notational conventions for propositional proof-contexts. (In particular, the notation "Γ[p:**H**] $\vdash_L$ c : C" would mean that the **H**-parameter [p : **H**] is not an element of Γ_h in the propositional context $\Gamma = \Gamma_h \smile \Gamma_\lambda$ and that it does not occur free in the formulas of Γ_λ.)

A logic *L* is said to be *proof-categorical* (alternatively: *L has a concept of a proof*) if there is a type-theoretic presentation [*L*] of *L* such that

$$(\Gamma \vdash_L a : A_1) \;\&\; (\Gamma \vdash_L a : A_2) \Rightarrow (A_1 \equiv A_2),$$

relative to [*L*] (where $\equiv$ stands for syntactic identity). Type-theoretically, this requirement is known as *unicity of typing* ([**UT**], for short). Intuitively, for any such an *L*, [**UT**-*L*] means that an *L*-proof is the proof of a *single* proposition.

A *proof-equation of L* is a statement of the form a $=_L$ b, where $\Gamma \vdash_L$ a,b for some Γ. For proof-categorical logics *L*, [*L*] *presents the L-proofs as a category of objects*. In other words, if $a_1 =_L a_2$ then, indeed, $\Gamma \vdash_L a_1$: A, $\Gamma \vdash_L a_2$: A, for some proof-context Γ and some A, with $\Vdash_L$ A :: **H**. Moreover, for each A, with $\Vdash_L$ A :: **H**, the pair $\tau_L = <\vdash_L, =_L>$ specifies the proofs of A as a sub-category of *L*-proofs. The proof-equations of a given logic *L* are shown by displaying the relevant proof-context and the type-information, i.e., the "proven proposition", as, e.g., in "$\Gamma \vdash_L \mathrm{a}_1 =_L \mathrm{a}_2 : \mathrm{A}$". Usually, the equational proof-system of some $[L]$ can be generated from the effective specification of a *notion of proof-reduction*.[8]

Proof-operators and proof-terms. The *proof-syntax* of a first-order *logic* L is thus obtained by generating the syntactic category $\Vdash_L(\Lambda)$ of *proof-terms in* L (*p-terms*, for short). The p-terms are generated inductively from a denumerable set Var_λ of Λ-atoms or *proof-variables* (*p-variables*, for short) x, y, z, ..., (possibly decorated), and, in specific first-order theories, based on L, a fixed set of *proof-constants* (*p-constants*), by closing under several *proof-operators*. So, the non-atomic proof-terms are *proof-forms* associated resp. to appropriate *operator-forms*. The syntactic (term-) forms can be considered as *means of codifying the output of the operator-forms*.

In general, a proof-operator in L can be viewed as an *abstraction operator acting on assumption forms* $\Gamma \vdash_L \mathrm{c} : \mathrm{C}$ and/or $\Gamma \Vdash_L \mathbf{t} :: \mathbf{U}$, resp. i.e., as *a partial map* $\mathbf{R}$ *from finite sets of assumption forms to assumption forms*. Here, the domain of $\mathbf{R}$ has elements of the form $\Gamma \models_L \varphi$, where $\varphi \equiv [\mathrm{c} : \mathrm{C}]$ or $\varphi \equiv [\mathbf{t} :: \mathbf{U}]$, with $\models_L$ used ambiguously for $\vdash_L$ and/or $\Vdash_L$. Conveniently, the range of a proof-operator $\mathbf{R}$ (i.e., its output) can be restricted to (proper) proof-statements $\Gamma \vdash_L \varphi$ (with $\varphi \equiv [\mathrm{c} : \mathrm{C}]$). Each proof-operator $\mathbf{R}$ can act *either* as a *proper abstractor* (for short, in absence of a better term: a *sumptor*) *or* as a *selector*.

> The *sumptors* should correspond, more or less, to so-called "introduction" rules in N-style ("natural deduction"-like) formulations of first-order logics, whereas the *selectors* are general *application forms* and cover the action of the familiar N-style "elimination" rules. It is, however, more appropriate to think of a proof-operator as being an *introduction-rule* for a specific proof-notation, i.e., a stipulation concerning the *admissible use* of a given proof-form ("proof-term"), whereas only the stipulations concerning proof-reduction processes ("détour eliminations", etc.) might count as proper "elimination" rules. Otherwise, the latter point of view is implicit in the Gentzen L-style ("sequent") proof-systems, where the proper derivation rules are viewed as "introduction" rules.

In the present setting, the meaning of a proof-operator is given unambiguously by a *proper derivation rule* (think, e.g., in terms of the old paradigm: "functions are [given by] rules"). Indeed, if, for $1 \leq i \leq n$, $\Gamma_i\Sigma_i \models \varphi_i$ (written also $\Gamma_i\Sigma_i.\varphi_i$, with $\varphi_i \equiv [\mathrm{c}_i : \mathrm{C}_i]$ or $\varphi_i \equiv [\mathbf{t}_i :: \mathbf{U}]$) are *assumption forms*, Γ_i, Σ_i are proof-contexts and $[\mathbf{R}](\Sigma_1.\varphi_1,\ldots,\Sigma_n.\varphi_n)$ is the *proof-form* (i.e., the "proof-term") introduced by the proof-operator $\mathbf{R}$, the *rule-form* of $\mathbf{R}$ is

$$(\mathbf{R}_\Rightarrow)\ \ \Gamma_i\Sigma_i \models \varphi_i,\ [1 \leq i \leq n] \Rightarrow \Gamma_1\ldots\Gamma_i\ldots\Gamma_n \vdash [\mathbf{R}](\Sigma_1.\varphi_1,\ldots,\Sigma_i.\varphi_i,\ldots,\Sigma_n.\varphi_n) : \mathrm{C},$$

while its *operator-form*

$$(\mathbf{R}_\lambda)\ \ \mathbf{R}(\Gamma_1\Sigma_1.\varphi_1,\ldots,\Gamma_n\Sigma_n.\varphi_n) = \Gamma_1\ldots\Gamma_n.[\mathbf{R}](\Sigma_1.\varphi_1,\ldots,\Sigma_n.\varphi_n) : \mathrm{C},$$

is a compact way of referring to the rule-form, and says that $\Gamma_1\ldots\Gamma_n.[\mathbf{R}](\Sigma_1.\varphi_1,\ldots,\Sigma_n.\varphi_n) : \mathrm{C}$ is *the value of* $\mathbf{R}$ *at* the set $\{\ \Gamma_i\Sigma_i \models \varphi_i : 1 \leq i \leq n\ \}$. Here, the Γ_i's are the *global* [*context-*] *parameters* of $\mathbf{R}$, whereas the Σ_i's are the *local* [*context-*] *parameters of* $\mathbf{R}$.

Both $(\mathbf{R}_\Rightarrow)$ and $(\mathbf{R}_\lambda)$ give the *general form* of a proof-operator $\mathbf{R}$. This can be specialized to a so-called *mono-parametric form*. The mono-parametric form of $\mathbf{R}$ is obtained from its general form *by forcing an identification of global context-parameters*. In other words, the mono-parametric variant of $(\mathbf{R}_\Rightarrow)$ is given by

$$(\mathbf{R}_{[\Rightarrow]})\ \ \Gamma\Sigma_i \models \varphi_i,\ [1 \leq i \leq n] \Rightarrow \Gamma \vdash [\mathbf{R}](\Sigma_1.\varphi_1,\ldots,\Sigma_i.\varphi_i,\ldots,\Sigma_n.\varphi_n) : \mathrm{C},$$

i.e., by a proof-operator with a single global context-parameter Γ which can be expressed by

$$(\mathbf{R}_{[\lambda]})\ \ \mathbf{R}(\Gamma\Sigma_1.\varphi_1,\ldots,\Gamma\Sigma_n.\varphi_n) = \Gamma.[\mathbf{R}](\Sigma_1.\varphi_1,\ldots,\ \Sigma_n.\varphi_n) : \mathrm{C}.$$

In the case of most familiar (monoidal) logics (as, e.g., the classical, intuitionistic, "minimal" logics, etc.), the mono-parametric proof-operators have the same strength as the general forms. This distinction becomes

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[8] In logic, a notion of proof-reduction is the formal counterpart of the intuitive concept of a *proof-détour elimination* (cf. [Rezuş 90]).

effective only in logics where some "structural" rules (as, e.g., the so-called "contraction" rules) are absent or are drastically restricted.

Concepts of consistency. The type-theoretic presentation of a logic yields two *natural* consistency concepts: *stratification consistency* (or *inferential consistency*) and *proof-consistency*. A logic L is said to be *stratification-consistent* (*inferentially consistent*) *relative to* a type-theoretic presentation $[L]$, if it is not the case that $[\,] \vdash_L \mathrm{a} : \bot$ for some a ($\Vdash_L \mathrm{a} :: \Lambda$). A proof-categorical logic L is *proof-consistent* relative to $[L]$ if it is not the case that $\Gamma \vdash_L \mathrm{a}_1 = \mathrm{a}_2 : \mathrm{A}$, for any two L-proofs a_1, a_2 of A ($\Gamma \vdash_L \mathrm{a}_1 : \mathrm{A}$, $\Gamma \vdash_L \mathrm{a}_2 : \mathrm{A}$). Given a logic L admitting of a type-theoretic presentation $[L]$, stratification-consistency relative to $[L]$ means that the standard proof-concept of L is correctly formalized *via* $[L]$, whereas proof-consistency relative to $[L]$ means that the category of L-proofs thereby isolated is non-trivial and that so are the sub-categories of objects associated to the "true propositions" of L.

The proof-context rules. Beyond the set of proper derivation rules (defining its proof-operators), any (first-order) logic requires specific *proof-context rules*, meant to state the *proof-context transformations admissible in* (the proofs of) L. The proof-context rules of a logic L admitting of a type-theoretic presentation are special (proof-) operators. These operators act, essentially, on the proof-context parts Γ of the proof-statements $\Gamma \vdash_L \varphi$ ($\varphi \equiv [\mathrm{c} : \mathrm{C}]$).

One can distinguish among *structural* context-rules, affecting the *global* context-parameters of a proper proof-operator, and so-called *"cut" rules*, which take also into account the free variables of φ within proofs of the form $\Gamma \vdash_L \varphi$, with $\varphi \equiv [\mathrm{c} : \mathrm{C}]$, and are meant to define the global *régime* of the substitution operators for **U**-and/or p-variables in proof-terms. For monoidal logics L admitting of a type-theoretic presentation (i.e., for logics with a non-composite consequence relation), we can state the proof-context rules beforehand, without reference to the specific (type- and/or proof-term-) structure of L. It is convenient to do this first for the case where the proof-contexts are thought of as being *sequences* (*of assumptions*). The rules for the *contexts-as-*[*assumption*]*-sets* view can be then obtained from this setting by leaving out the specific information ("context-sensitive", so-to-speak), meant to handle the *sequential* features of the proof-context representation.

Let $\varphi \equiv [\mathrm{c} : \mathrm{C}]$, for some c and C, with $\Vdash_L \mathrm{c} :: \Lambda$ and $\Vdash_L \mathrm{C} :: \mathbf{H}$. A generic (first-order) proof-context is denoted by $\Gamma = \Gamma_u \smile \Gamma_\lambda$. What follows fits the description of proof-context behaviors in *first-order monoidal logics*. Since the definition is schematic, we omit the L-subscripts.

Definition (*"Sequential" proof-context rules*).

1.1 "Structural" rules.

$<I>$ $[\mathrm{u}_1{:}\mathbf{U}]\ldots[\mathrm{u}_m{:}\mathbf{U}] \smile [\mathrm{x}{:}\mathrm{A}] \vdash \mathrm{x} : \mathrm{A}$, if $\{\ \mathrm{u}_1, \ldots, \mathrm{u}_m\ \} = \mathrm{FV}_u(\mathrm{A})$,

$<K>$ $\Gamma \vdash \varphi \Rightarrow \Gamma[\mathrm{x}{:}\mathrm{A}] \vdash \varphi$,

$<KW>$ $\Gamma[\mathrm{x}{:}\mathrm{A}] \vdash \varphi \Rightarrow \Gamma[\mathrm{x}{:}\mathrm{A}][\mathrm{x}{:}\mathrm{A}] \vdash \varphi$,

$<W>$ $\Gamma[\mathrm{x}{:}\mathrm{A}][\mathrm{x}{:}\mathrm{A}] \vdash \varphi\llbracket\mathrm{x}\rrbracket \Rightarrow \Gamma[\mathrm{x}{:}\mathrm{A}] \vdash \varphi\llbracket\mathrm{x}\rrbracket$,

$<C>$ $\Gamma[\mathrm{x}{:}\mathrm{A}][\mathrm{y}{:}\mathrm{B}] \vdash \varphi\llbracket\mathrm{x,y}\rrbracket \Rightarrow \Gamma[\mathrm{y}{:}\mathrm{B}][\mathrm{x}{:}\mathrm{A}] \vdash \varphi\llbracket\mathrm{x,y}\rrbracket$,

$<K_u>$ $\Gamma_u \smile \Gamma_\lambda \vdash \varphi \Rightarrow \Gamma_u[\mathrm{u}{:}\mathbf{U}] \smile \Gamma_\lambda \vdash \varphi$,

$<KW_u>$ $\Gamma_u[\mathrm{u}{:}\mathbf{U}] \smile \Gamma_\lambda \vdash \varphi\llbracket\mathrm{u}\rrbracket \Rightarrow \Gamma_u[\mathrm{u}{:}\mathbf{U}][\mathrm{u}{:}\mathbf{U}] \smile \Gamma_\lambda \vdash \varphi\llbracket\mathrm{u}\rrbracket$,

$<W_u>$ $\Gamma_u[\mathrm{u}{:}\mathbf{U}][\mathrm{u}{:}\mathbf{U}] \smile \Gamma_\lambda \vdash \varphi\llbracket\mathrm{u}\rrbracket \Rightarrow \Gamma_u[\mathrm{u}{:}\mathbf{U}] \smile \Gamma \vdash \varphi\llbracket\mathrm{u}\rrbracket$,

$<C_u>$ $\Gamma_u[\mathrm{u}{:}\mathbf{U}][\mathrm{v}{:}\mathbf{U}] \smile \Gamma_\lambda \vdash \varphi\llbracket\mathrm{u,v}\rrbracket \Rightarrow \Gamma_u[\mathrm{v}{:}\mathbf{U}][\mathrm{u}{:}\mathbf{U}] \smile \Gamma_\lambda \vdash \varphi\llbracket\mathrm{u,v}\rrbracket$.

1.2 "Cut"-rules.

$<\$K>$ $\Gamma[\mathrm{x}{:}\mathrm{A}] \vdash \varphi \Rightarrow \Gamma \vdash \varphi$, if x is not free in φ,

$<\$W>$ $\Gamma[\mathrm{x}{:}\mathrm{A}][\mathrm{y}{:}\mathrm{A}] \vdash \varphi\llbracket\mathrm{x,y}\rrbracket \Rightarrow \Gamma[\mathrm{x}{:}\mathrm{A}][\mathrm{x}{:}\mathrm{A}] \vdash \varphi\llbracket\mathrm{x,x}\rrbracket$,

$<\$_uK>$ $\Gamma_u[\mathrm{u}{:}\mathbf{U}] \smile \Gamma_\lambda \vdash \varphi \Rightarrow \Gamma_u \smile \Gamma_\lambda \vdash \varphi$, if u is not free in φ,

$<\$_uW>$ $\Gamma_u[\mathrm{u}{:}\mathbf{U}][\mathrm{v}{:}\mathbf{U}] \smile \Gamma_\lambda \vdash \varphi\llbracket\mathrm{u,v}\rrbracket \Rightarrow \Gamma_u[\mathrm{u}{:}\mathbf{U}][\mathrm{u}{:}\mathbf{U}] \smile \Gamma_\lambda \vdash \varphi\llbracket\mathrm{u,u}\rrbracket$,

$< \$ >$ $\Gamma_1 \vdash$ a : A, Γ[x:A] $\vdash \varphi[\![$x$]\!]$ $\Rightarrow \Gamma\Gamma_1 \vdash \varphi[\![$x:=a$]\!]$,
$< \$_{[u]} >$ $\Gamma_u \Vdash$ **t** :: **U**, $\Gamma[\![$u:**U**$]\!] \vdash \varphi[\![$u$]\!]$ $\Rightarrow \Gamma_u(\Gamma[\![$u:=**t**$]\!]) \vdash \varphi$ $[\![$u:=**t**$]\!]$.

For most familiar logics, this is, in fact, very redundant. Some remarks are in order.

(1) The \$-rules are so-called "cut"-rules and – as noted above – are intended, in general, to establish a purely formal *régime* for the manipulation of the *substitution operators* for **U**- and proof-variables (in **U**-terms, formulas and/or proof-terms).[9]

(2) The "cut"-rules for **U**-variables (the $\$_u$-rules) are usually ignored in logic books [they are rather trivial, indeed]. In particular, if $\Gamma[\![$u:**U**$]\!] = \Gamma_1$[u:**U**], where u is not in Γ_1, the "cut"-rule $< \$_{[u]} >$ becomes

$< \$_u >$ $\Gamma_u \Vdash$ **t** :: **U**, Γ[u:**U**] $\vdash \varphi[\![$u$]\!]$ $\Rightarrow \Gamma_u\Gamma \vdash \varphi[\![$u:=**t**$]\!]$.

(3) It is easy to see that, *ceteris paribus*, the "cut"-rules $< \$W >$ and $< \$_uW >$ are special cases of the general "cuts" $< \$ >$ and $< \$_{[u]} >$.

(4) The "weakening reversals" $< \$K >$ and $< \$_uK >$ can be shown to be *admissible* (in the sense of "admissible *rule*", *à la* P. Lorenzen and H. B. Curry) for most logics (among which those of concern below).

(5) The rules $< KW >$ and $< KW_u >$ are inverses of $< W >$ and $< W_u >$, resp. but, clearly, the duplications $< K >$, $< KW >$ and $< K_u >$, $< KW_u >$ are due here to a notational accident, i.e., to the fact that we have decided that "Γ[x:A]" (resp. "Γ[u:**U**]") means that [x:A] (resp. [u:**U**]) is *not* in Γ (this is meant to simplify the statement of proof-context conditions in proper derivation rules).

(6) *Ceteris paribus*, the joint effect of $< K >$, $< I >$ can be obtained, in normal cases, by a "global context-projection" rule (as, e.g., in *Classical Automath*, which is, essentially, a *Minimalkalkül*-like system):

$< K\Gamma >$ $\Gamma \vdash$ x : A, if [x : A] is in Γ.

Proof-contexts as assumption-sets. If the proof-contexts are thought of as (assumption-) *sets*, the "structural" rules $< KW >$, $< W >$, $< C >$, $< KW_u >$, $< W_u >$, $< C_u >$ are redundant, while the "cut"-rules $< \$W >$, $< \$_uW >$ become, resp.

$< \$W >$ Γ[x:A][y:A] $\vdash \varphi[\![$x,y$]\!]$ $\Rightarrow \Gamma$[x:A] $\vdash \varphi[\![$x,x$]\!]$,
$< \$_uW >$ Γ_u[u:**U**][v:**U**] $\smile \Gamma_\lambda \vdash \varphi[\![u,v]\!]$ $\Rightarrow \Gamma_u$[u:**U**] $\smile \Gamma_\lambda \vdash \varphi[\![u,u]\!]$.

In detail, for the case of the *proof-contexts-as-*[*assumption*]*-sets* view, we should be also satisfied with the following compact set of proof-context rules:

Definition (*Proof-context* [= "*assumption set*"] *rules*).

1.1 "Structural" rules.

$< I >$ [u_1:**U**]... [u_m:**U**] $\smile$ [x:A] $\vdash$ x : A, if { $u_1,\ldots,u_m$ } = FV_u(A),
$< K >$ $\Gamma \vdash \varphi$ $\Rightarrow \Gamma$[x:A] $\vdash \varphi$,
$< K_u >$ $\Gamma_u \smile \Gamma_\lambda \vdash \varphi$ $\Rightarrow \Gamma_u$[u:**U**] $\smile \Gamma_\lambda \vdash \varphi$.

1.2 "Cut"-rules.

$< \$ >$ $\Gamma_1 \vdash$ a : A, Γ[x:A] $\vdash \varphi[\![$x$]\!]$ $\Rightarrow \Gamma\Gamma_1 \vdash \varphi[\![$x:=a$]\!]$,
$< \$_{[u]} >$ $\Gamma_u \Vdash$ **t** :: **U**, $\Gamma[\![$u:**U**$]\!] \vdash \varphi[\![$u$]\!]$ $\Rightarrow \Gamma_u(\Gamma[\![$u:=**t**$]\!]) \vdash \varphi[\![$u:=**t**$]\!]$,

where the addition of (the admissible "cut" rules) $< \$K >$ and $< \$_uK >$ is *facultative*. As earlier, we can also replace $< \$_{[u]} >$, in this setting, by its special case:

$< \$_u >$ $\Gamma_u \Vdash$ **t** :: **U**, Γ[u:**U**] $\vdash \varphi[\![$u$]\!]$ $\Rightarrow \Gamma_u\Gamma \vdash \varphi[\![$u:=**t**$]\!]$,

whereas, for the classical, intuitionistic, "minimal", etc. logics, one can also replace $< I >$ and $< K >$ by $< K\Gamma >$.

[9]Here, one has to deal with "formal (symbolic) manipulations as effected by *any* (symbolic) agent, no matter whether the latter one is human or mechanistic in nature".

Unlike the above, the proof-context rules for (first-order) logics L with a composite consequence relation cannot be stated globally: their exact form depends on the (*local*) *proof-operator structure* of L. (See, e.g., *mutatis mutandis*, [Belnap 82].)

Chapter II

Proof-syntax

Classical logic and sub-systems: the type-structure. We rely next on type-structures [⊥,→,(∧),∀], suited for the first-order classical logic **CQ**, and [⊥,→,∧,∨,∀,∃], suited for *Minimalkalkül* **MQ**, the Curry logic **DQ**, the Heyting logic **HQ**, etc. For heuristic purposes, the structure [⊥,→,∧,∨,∀,∃] is also considered next as an alternative type-structure for **CQ**.

The spelling of (*first-order*) *formulas*. We spare on parentheses as usually, by assuming associativity to the left, omitting the outermost pair, and by applying familiar conventions concerning priority in parsing by dot-separators. So A → B → A → A stands for (((A → B) → A) → A), and, with dot-separators, A → .A → B → B stands for (A → ((A → B) → B)).

Definitions. For first-order classical logic **CQ**, as formulated relative to a type-structure [⊥,→,∧,∀], the following are taken as *standard abbreviations*:

- (¬A) := (A → ⊥) [*inferential negation*],
- (⊤) := ¬⊥ [≡ (⊥ → ⊥)] [*internal verum*],
- (A ∨ B) := ¬(¬A ∧ ¬B) [*disjunction*],
- (A ↔ B) := (A → B) ∧ (B → A) [*equivalence*],
- (∃u.A⟦u⟧) := ¬(∀u.(¬A⟦u⟧)) [*existence*].

Remark (*Definable type-constructors: proof-theoretic relevance*). For the proof-theory of **CQ** (formulated relative to [⊥,→,∧,∀]-primitives), the choice of these definitions is *not arbitrary*. Indeed, we must be able to associate *derived proof-operations* to a definable type-constructor in order to use it in a proof- theoretically relevant way (see details below). In general, it is *not* the case that any definition that makes sense on a provability-level (truth-functionally, etc.) should be also useful/relevant on a proof-level. In the absence of a primitive ∧ (*conjunction*), one can define, in **CQ**, relative to [⊥,→,(∀)], "algebraic" type-constructors analogous to ∧ and ∨, still admitting of well-behaved proof-operations, as, for instance, *either*

- $(A \otimes_{\sqcap} B)$:= ¬(A → ¬B) [(*strong*) *internal conjunction*, default: $\otimes \equiv \otimes_{\sqcap}$],
- $(A \oplus_{\sqcap} B)$:= $\neg(\neg A \otimes_{\sqcap} \neg B)$ [(*strong*) *internal disjunction*, default: $\oplus \equiv \oplus_{\sqcap}$],

or

- (A ⟦ B) := (¬A → B) [*weak disjunction*],
- $(A \oplus_{\sqcup} B)$:= (A ⟦ ¬¬B) [*weak internal disjunction*],
- $(A \otimes_{\sqcup} B)$:= $\neg(\neg A \oplus_{\sqcup} \neg B)$ [*weak internal conjunction*],

or

- $(A \oplus_{\top} B)$:= (A → B → B) [(*Tarski*) *inferential disjunction*],
- $(A \otimes_{\top} B)$:= $\neg(\neg A \oplus_{\top} \neg B)$ [(*Tarski*) *inferential conjunction*],

(*à la* Alfred Tarski[10]), etc. with, in each case (where ⊗ is used ambiguously), a simulated equivalence [*internal equivalence*]

- $(A \leftrightarrow_{\otimes} B)$:= (A → B) ⊗ (B → A).

Each one of the [⊗,⊕]-pairs above admits of "intensional" proof-operations. This means that the usual extensionality conditions for the associated proof-operators should fail; e.g., the pairings (pairs-*cum*-projections) that can be associated to any one of the ⊗'s won't be "surjective" (and analogously about extensionality properties for ⊕-operations). In the present setting, by the way, things like the "weak" disjunction ⟦ are *not productive proof-theoretically*.

[10]First proposed a logic seminar of Jan Łukasiewicz, ± 1921.

As expected, $\vee$, $\exists$ are taken as primitive notions in *Minimalkalkül* **MQ**, the "complete refutability logic" **DQ** of Curry and in Heyting's logic **HQ**. This notation is also used with subscript-decorations, whenever necessary (subscripts: m, d, h, resp.)

Syntactic notions. The syntactic concepts of a *subterm of a* **U**-*term*, *subformula*, *free/bound* **U**-*variable* (*occurring*) *in* a **U**-term (resp. formula) are supposed to be introduced in the usual way. In what follows, we use $\mathrm{FV}_u(\mathbf{t})$, $\mathrm{BV}_u(\mathbf{t})$, $\mathrm{FV}_u(\mathrm{A})$, $\mathrm{BV}_u(\mathrm{A})$ resp., in order to refer to the corresponding sets of free/bound variables. A **U**-term **t** (resp. a first-order formula A) is **U**-*closed* if $\mathrm{FV}_u(\mathbf{t}) = \emptyset$ (resp. $\mathrm{FV}_u(\mathrm{A}) = \emptyset$), else it is **U**-*open*. The possibility of relettering systematically the bound **U**-variables (α_u-*conversion*) of a provability language is assumed tacitly. Everywhere in the sequel, the "windows" (or *the display brackets*) $\llbracket \ldots \rrbracket$ do *not* belong to the object-syntax; they are supposed to display internal components. So, if ξ is a "free atom", "$\alpha\llbracket\xi\rrbracket$" means that ξ may occur free in the syntactic environment $\alpha\llbracket\ldots\rrbracket$. The uniform *substitution operators* for **U**-variables are shown by "$\alpha\llbracket \mathrm{u}{:=}\mathbf{t}\rrbracket$", where $\mathrm{u} \in \mathrm{Var}_u$, $\Vdash_L \mathbf{t} :: \mathbf{U}$ and $\Vdash_L \alpha :: \mathbf{U}$ or $\Vdash_L \alpha :: \mathbf{H}$, for any logic L of concern. They are supposed to be defined in the expected way.

Boolean proof-terms and proof-operators. The *formal proof-notation* for **CQ** is given by the following scheme, recording *the structure of the Boolean proof-operators* associated to a *primitive* type-structure of the form $[\bot,\to,\wedge,\forall]$.

Definition (*Boolean proof-terms/proof-operators for* $[\bot,\to,\wedge,\forall]$).

(1) A proof-variable is a proof-term.

(2) If a, b, e, f are proof-terms, **t** is a **U**-term and A, B, C are formulas then the following are proof-terms [resp. proof-operator forms]:

(2_1) *positive sumptors*:

- $\lambda\mathrm{x{:}A.b}\llbracket\mathrm{x}\rrbracket \quad [\equiv \lambda_{\vdash}([\mathrm{x{:}A}].\mathrm{b}\llbracket\mathrm{x}\rrbracket{:}\mathrm{B})]$,
- $<\mathrm{a{:}A,b{:}B}> \quad [\equiv \lambda_{\natural}(\mathrm{a{:}A,b{:}B})]$,
- $!\mathrm{u.a}\llbracket\mathrm{u}\rrbracket \quad [\equiv \lambda_{\cup}([\mathrm{u}{:}\mathbf{U}].\mathrm{a}\llbracket\mathrm{u}\rrbracket{:}\mathrm{A}\llbracket\mathrm{u}\rrbracket)]$,

(2_2) *the "prime" negative sumptors*: where A is a "prime" formula,

- $\gamma\mathrm{x}{:}\neg\mathrm{A.e}\llbracket\mathrm{x}\rrbracket \quad [\equiv \gamma_{\vdash}([\mathrm{x}{:}\neg\mathrm{A}].\mathrm{e}\llbracket\mathrm{x}\rrbracket{:}\bot)]$,

(2_3) "*application forms*" (*special positive selectors*):

- $\mathrm{f(a)} \quad [\equiv @_{\vdash}(\mathrm{f{:}A{\to}B,a{:}A})]$,
- $\mathbf{p}_1(\mathrm{f{:}A{\wedge}B}) \quad [\equiv @^{\natural}_1(\mathrm{f{:}A{\wedge}B})]$,
- $\mathbf{p}_2(\mathrm{f{:}A{\wedge}B}) \quad [\equiv @^{\natural}_2(\mathrm{f{:}A{\wedge}B})]$,
- $\mathrm{f}[\mathbf{t}] \quad [\equiv @_{\cup}(\mathrm{f}{:}\forall\mathrm{u.A}\llbracket\mathrm{u}\rrbracket,\mathbf{t}{:}\mathbf{U})]$.

Here, the "negative" sumptors (γ-abstractions) of the form $\gamma\mathrm{x}{:}\neg\mathrm{C.e}\llbracket\mathrm{x}\rrbracket$ are defined only for "prime" formulas C. From this, the full $\lambda\gamma$-(proof)-syntax can be obtained by setting, inductively, for $\mathrm{C} \equiv \bot$, $(\mathrm{A{\to}B})$, $(\mathrm{A{\wedge}B})$, $\forall\mathrm{u.A}\llbracket\mathrm{u}\rrbracket$, resp.,

Definition (*General* γ-*abstraction for* $[\bot,\to,\wedge,\forall]$-*structures*).

$$
\begin{array}{lll}
[\gamma\bot]: & \gamma\mathrm{x}{:}\top.\mathrm{e}\llbracket\mathrm{x}\rrbracket & := \mathrm{e}\llbracket\mathrm{x}{:=}\Omega\rrbracket, \textbf{ where } \Omega \equiv \lambda\mathrm{x}{:}\bot.\mathrm{x}, \\
[\gamma{\to}]: & \gamma\mathrm{x}{:}\neg(\mathrm{A{\to}B}).\mathrm{e}\llbracket\mathrm{x}\rrbracket & := \lambda\mathrm{x}_0{:}\mathrm{A}.\gamma\mathrm{x}_1{:}\neg\mathrm{B.e}\llbracket\mathrm{x}{:=}\lambda\mathrm{z}{:}(\mathrm{A{\to}B}).\mathrm{x}_1(\mathrm{z}(\mathrm{x}_0))\rrbracket, \\
[\gamma\wedge]: & \gamma\mathrm{x}{:}\neg(\mathrm{A{\wedge}B}).\mathrm{e}\llbracket\mathrm{x}\rrbracket & := <\mathrm{a} : \mathrm{A},\ \mathrm{b} : \mathrm{B}>, \textbf{ where} \\
 & & \mathrm{a} \equiv \gamma\mathrm{x}_1{:}\neg\mathrm{A.e}\llbracket\mathrm{x}{:=}\lambda\mathrm{z}{:}(\mathrm{A{\wedge}B}).\mathrm{x}_1(\mathbf{p}_1(\mathrm{z{:}A{\wedge}B}))\rrbracket, \textbf{ and} \\
 & & \mathrm{b} \equiv \gamma\mathrm{x}_2{:}\neg\mathrm{B.e}\llbracket\mathrm{x}{:=}\lambda\mathrm{z}{:}(\mathrm{A{\wedge}B}).\mathrm{x}_2(\mathbf{p}_2(\mathrm{z{:}A{\wedge}B}))\rrbracket, \\
[\gamma\forall]: & \gamma\mathrm{x}{:}\neg(\forall\mathrm{u.A}).\mathrm{e}\llbracket\mathrm{x}\rrbracket & := !\mathrm{v}.\gamma\mathrm{x}_1{:}\neg\mathrm{A.e}\llbracket\mathrm{x}{:=}\lambda\mathrm{z}{:}(\forall\mathrm{u.A}).\mathrm{x}_1(\mathrm{z}[\mathrm{v}])\rrbracket,\ [\mathrm{u} \not\equiv \mathrm{v}],
\end{array}
$$

such that the proof-variables x_0, x_1, x_2, resp. and the **U**-variable u are *fresh for* $\mathrm{e}\llbracket\mathrm{x}\rrbracket$.[11]

[11] This *proviso* can be stated in more rigorous terms, of course, as soon as we are able to identify formally the *bound* and *free* p-*variables* of a proof-term; see below.

In what follows, the primitive proof-term stratification rules and the equational postulates of **CQ** will be stated, conveniently, for "prime" γ-abstractions, whereas the corresponding general forms are shown to be derivable. As usually, we write (fa) for f(a), omitting oft the outermost pair of parentheses. Alternatively, one can also write $\mathbf{p}_i\llbracket A,B\rrbracket(f)$, or just $\mathbf{p}_i(f)$, for $\mathbf{p}_i(f{:}A\wedge B)$ [i := 1,2], as well as $<a,b>$ for $<a{:}A,b{:}B>$, if no confusions are likely to occur.

> Putting the γ's aside, this brings us back to a rather *standard* λ-calculus notation, with perhaps, as only idiosyncrasy, the use of a familiar ! [$= \lambda_\cup$], at a *proof*-level, for the **U**-generalization operation [mnemonics: "(λ)-abstraction over **U**" or yet "(functional) abstraction over a fixed universe (of individuals)"]. Casually, other people have got a bare λ instead, or an upper-case Λ, or yet a $\bigwedge$, in slightly different a context: we have better uses for the latter three symbols, already rather overloaded. Otherwise, the basic notational habits are here those common in *mathematical* practice, so that the logician's use-and-mention *etiquette* is nearly always ignored: this kind of shorthand is not known to generate confusion by itself. . .

Remark (*Extensional pairing in* [**M,H,C**]**Q**). In a Boolean setting based on at least [$\bot,\rightarrow$], the conjunction and the associated proof-operators (pairs-*cum*-projections) count as redundant (and analogously for disjunction). This is not exactly the case, once we are also interested in the *equational behaviors* of these operators. As is well-known from the ordinary ("simple") typed λ-calculus, the equational conditions defining an extensional ("surjective") pairing cannot be simulated in purely inferential terms[12],whereas the addition of the "negative" sumptor γ does not (and *can* not) change this situation.

Remark (*General positive selectors and "appplication forms"*). In the above, the *standard "application forms"* (i.e., the so-called *"functional" application*, the *projections* and the *instantiation*, resp.) are familiar constructs implicit in "natural deduction" (N-style) systems of logic. They can be obtained as special cases of *the general positive selectors* (proof-term forms [resp. proof-operator forms]):

- $\bigwedge_{\vdash}(y{:}B).c\llbracket y\rrbracket \Diamond f(a)$ $\quad[\equiv \bigwedge_{\vdash}([y{:}B].c\llbracket y\rrbracket{:}C, f{:}A\rightarrow B, a{:}A)]$,
- $\bigwedge_{\natural}(x{:}A,y{:}B).c\llbracket x,y\rrbracket \Diamond f$ $\quad[\equiv \bigwedge_{\natural}([x{:}A][y{:}B].c\llbracket x,y\rrbracket{:}C, f{:}A\wedge B)]$,
- $\bigwedge_{\cup}(x{:}A\llbracket \mathbf{t}\rrbracket).c\llbracket x\rrbracket \Diamond f[\mathbf{t}]$ $\quad[\equiv \bigwedge_{\cup}([x{:}A\llbracket \mathbf{t}\rrbracket]).c\llbracket y\rrbracket{:}C, f{:}\forall u.A, \mathbf{t}{:}\mathbf{U})]$.

Indeed, we have – with the latter three taken as primitives, in place of the "standard application forms" – the following *definitions* (proof-term forms [resp. proof-operator forms]):

- $f(a)$ $\quad[\equiv @_{\vdash}(f{:}A\rightarrow B, a{:}A)]$ $\quad:= \bigwedge_{\vdash}(y{:}B).y \Diamond f(a)$,
- $\mathbf{p}_1(f{:}A\wedge B)$ $\quad[\equiv @^{\natural}_1(f{:}A\wedge B)]$ $\quad:= \bigwedge_{\natural}([x{:}A][y{:}B]).x \Diamond f)$,
- $\mathbf{p}_2(f{:}A\wedge B)$ $\quad[\equiv @^{\natural}_2(f{:}A\wedge B)]$ $\quad:= \bigwedge_{\natural}([x{:}A][y{:}B]).y \Diamond f)$,
- $f[\mathbf{t}]$ $\quad[\equiv @_{\cup}(f{:}\forall u.A\llbracket u\rrbracket, \mathbf{t}{:}\mathbf{U})]$ $\quad:= \bigwedge_{\cup}(x{:}A\llbracket \mathbf{t}\rrbracket).x \Diamond f[\mathbf{t}])$.

Conversely, *the general positive selectors* are definable, in **CQ**, **MQ**, **HQ**, etc., from the standard "application forms", by uniform substitution, viz. by

- $\bigwedge_{\vdash}(y{:}B).c\llbracket y\rrbracket \Diamond f(a)$ $\quad:= c\llbracket y{:=}f(a)\rrbracket$,
- $\bigwedge_{\natural}(x{:}A,y{:}B).c\llbracket x,y\rrbracket \Diamond f$ $\quad:= c\llbracket x{:=}\mathbf{p}_1(f{:}A\wedge B)\rrbracket\llbracket y{:=}\mathbf{p}_2(f{:}A\wedge B)\rrbracket$,
- $\bigwedge_{\cup}(x{:}A\llbracket \mathbf{t}\rrbracket).c\llbracket x\rrbracket \Diamond f[\mathbf{t}]$ $\quad:= c\llbracket y{:=}f[\mathbf{t}]\rrbracket$.

Finally, in *specific first-order theories* based on **CQ**, the *proof-constants* are proof-terms, too. As expected, a proof-constant stands always for a given – supposedly known – *primitive* proof, e.g., for *the evidence behind* [= "the proof of"] *some axiom*.[13]

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[12] This follows from an undefinability result in *type-free* λ-calculus, due to Henk Barendregt [74]. The equational behavior of the "intensional" algebraic proof-operators is discussed later.

[13] Cf. with the "primitive notions" (as expressed by the so-called PN-*lines*) of most *Automath* systems. For the sake of completeness, we may choose to represent such "notions" schematically, by an Ω-symbol, say, while using type-structures with

[*Other*] *negative proof-operators*. In a genuinely Boolean setting, with *standard* [i.e., Ockham/De Morgan] *definitions* for $A \vee B$ [$\equiv_{df}$ $\neg(\neg A \wedge \neg B)$] and $\exists u.A\llbracket u\rrbracket$ [$\equiv_{df}$ $\neg(\forall u.\neg A\llbracket u\rrbracket)$], one can define *the proof-operators associated* to (the Boolean) $\vee$, $\exists$, resp., viz. the *negative sumptors* $\mathbf{j}$, $\mathbf{J}$ and the *negative selectors* $\bigvee_\natural$, $\bigvee_\cup$, resp. Indeed, the following definitions (using positive proof-operators and γ-abstractions) preserve the extensional proof-behaviors of the Boolean [$\vee$,$\exists$]-proof-operators.

Definition (*Derived negative Boolean* [$\vee$,$\exists$]*-proof-operators*).

(1) *Boolean* $\vee$*-proof-operators*:
- $\mathbf{j}(x{:}\neg A, y{:}\neg B).e\llbracket x,y\rrbracket$ $:= \lambda z{:}(\neg A \wedge \neg B).e\llbracket x{:=}\mathbf{p}_1(z)\rrbracket\llbracket y{:=}\mathbf{p}_2(z)\rrbracket$,
- $\bigvee_\natural(z{:}\neg C).f \Diamond [\lambda x{:}A.e_1\llbracket x,z\rrbracket, \lambda y{:}B.e_2\llbracket y,z\rrbracket]$ $:= \gamma z{:}\neg C.f(<\lambda x{:}A.e_1\llbracket x,z\rrbracket, \lambda y{:}B.e_2\llbracket y,z\rrbracket>)$.

(2) *Boolean* $\exists$*-proof-operators*:
- $\mathbf{J}(x{:}\neg A\llbracket \mathbf{t}\rrbracket).e\llbracket x\rrbracket$ $:= \lambda z{:}(\forall u.\neg A\llbracket u\rrbracket).e\llbracket x{:=}z[\mathbf{t}]\rrbracket$,
- $\bigvee_\cup(z{:}\neg C).f \Diamond [!u.\lambda x{:}A\llbracket u\rrbracket.e\llbracket u,x,z\rrbracket]$ $:= \gamma z{:}\neg C.f(!u.\lambda x{:}A\llbracket u\rrbracket.e\llbracket u,x,z\rrbracket)$.

If we are relying on a type-structure [$\bot,\to,\wedge,\vee,\forall,\exists$], with $\vee$ and $\exists$ among the primitives, we must add also the negative sumptors and the negative selectors to the definition of Boolean proof-terms/operators. So, although redundant, the most general scheme that can be used for the introduction of the proof-operators of **CQ**, relative to [$\bot,\to,\wedge,\vee,\forall,\exists$], should be the following one:

Definition (*Boolean proof-terms/operators for* [$\bot,\to,\wedge,\vee,\forall,\exists$]).

(1) A proof-variable is a proof-term.

(2) If a, b, c, e, f are proof-terms, $\mathbf{t}$ is a $\mathbf{U}$-term and A, B, C are formulas then the following are proof-terms (proof-term forms [resp. proof-operator forms]):

(2_1) *positive sumptors*:
- $\lambda x{:}A.b\llbracket x\rrbracket$ [$\equiv \lambda_\vdash([x{:}A].b\llbracket x\rrbracket{:}B)$],
- $<a{:}A, b{:}B>$ [$\equiv \lambda_\natural(a{:}A, b{:}B)$],
- $!u.a\llbracket u\rrbracket$ [$\equiv \lambda_\cup([u{:}\mathbf{U}].a\llbracket u\rrbracket{:}A\llbracket u\rrbracket)$],

(2_2) *negative sumptors*:
- $\gamma x{:}\neg A.e\llbracket x\rrbracket$ [$\equiv \gamma_\vdash([x{:}\neg A].e\llbracket x\rrbracket{:}\bot)$], [A arbitrary: unrestricted $\gamma_\vdash$],
- $\mathbf{j}(x{:}\neg A, y{:}\neg B).e\llbracket x,y\rrbracket$ [$\equiv \gamma_\natural([x{:}\neg A][y{:}\neg B].e\llbracket x,y\rrbracket{:}\bot)$],
- $\mathbf{J}(x{:}\neg A\llbracket \mathbf{t}\rrbracket).e\llbracket x\rrbracket$ [$\equiv \gamma_\cup([x{:}\neg A\llbracket \mathbf{t}\rrbracket].e\llbracket x\rrbracket{:}\bot)$],

(2_3) (*general*) *positive selectors*:
- $\bigwedge_\vdash(y{:}B).c\llbracket y\rrbracket \Diamond f(a)$ [$\equiv \bigwedge_\vdash([y{:}B].c\llbracket y\rrbracket{:}C, f{:}A\to B, a{:}A)$],
- $\bigwedge_\natural(x{:}A, y{:}B).c\llbracket x,y\rrbracket \Diamond f$ [$\equiv \bigwedge_\natural([x{:}A][y{:}B].c\llbracket x,y\rrbracket{:}C, f{:}A\wedge B)$],
- $\bigwedge_\cup(x{:}A\llbracket \mathbf{t}\rrbracket).c\llbracket x\rrbracket \Diamond f[\mathbf{t}]$ [$\equiv \bigwedge_\cup([x{:}A\llbracket \mathbf{t}\rrbracket]).c\llbracket x\rrbracket{:}C, f{:}\forall u.A, \mathbf{t}{:}\mathbf{U})$].

(2_4) *negative selectors*:
- $\bigvee_\natural(z{:}\neg C).f \Diamond [\lambda x{:}A.e_1\llbracket x,z\rrbracket, \lambda y{:}B.e_2\llbracket y,z\rrbracket]$ [$\equiv \bigvee_\natural(f{:}A\vee B, [x{:}A][z{:}\neg C].e_1\llbracket x,z\rrbracket{:}\bot, [y{:}B][z{:}\neg C].e_2\llbracket y,z\rrbracket{:}\bot]$,
- $\bigvee_\cup(z{:}\neg C).f \Diamond [!u.\lambda x{:}A\llbracket u\rrbracket.e\llbracket u,x,z\rrbracket]$ [$\equiv \bigvee_\cup(f{:}\exists u.A\llbracket u\rrbracket, [u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket][z{:}\neg C].e\llbracket u,x,z\rrbracket{:}\bot)$].

The latter definition yields also, in a straightforward way, the proof-operator structure of an appropriate Gentzen *L*-style ("sequent") presentation for **CQ**.

Actually, most proof-operators encountered in the "sequent" proof-systems for **CQ**, **HQ**, **MQ**, etc. can be obtained from the above by *specializing* the *global* context-parametrization of the corresponding selectors. So, the present point of view amounts to a generalization of both the "natural deduction"

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a primitive $\top$-object [*verum*]: in this case, one needs also a special stratification stipulation to the effect that Ω "proves" $\top$, in any proof-context. Any attempt to set up *a priori* general stipulations on the *equational behavior* of Ω looks debatable, however. In what follows, we assume – in agreement with a wise tradition – that there is no such a thing like "the common [theoretical] content of all first-order theories" (beyond, perhaps, *formal logic*). See also [Rezuş 90] for the specifics of this approach.

and the "sequent" formulations of the proof-theory of familiar first-order logics (classical, Heyting, "minimal", etc.).

Remark (*Local γ-definability*). In the extended Boolean setting (based on [⊥,→,∧,∨,∀,∃]) the γ-abstractor is definable in terms of, say, ∨-proof-operators and *special instances* of it. This is analogous to the way *one can simulate intuitionistically local γ-operators* $\gamma_{f:\mathrm{A}}$ in terms of **HQ**-proof-operators, for each formula A in [⊥,→,∧,∨,∀,∃] and each (Heyting) proof-term f such that $\Gamma \vdash f : \neg_h \mathrm{A} \vee_h \mathrm{A}$ holds in the Heyting proof-calculus (cf. below).

Remark (*Intensional algebraic proof-operators in* **CQ**).

(1) The stratification properties of a [Boolean] pairing can be simulated with the means of a proof-structure based on [⊥,→] alone. Set, e.g., $\mathrm{A} \otimes \mathrm{B} := \neg(\mathrm{A} \to \neg\mathrm{B})$, as above [i.e., using $\otimes \equiv \otimes_\sqcap$], with associated proof-forms:

- $\prec$ a:A, b:B $\succ$ $[\equiv \lambda_\otimes(\mathrm{a{:}A,b{:}B})]$ $:= \lambda \mathrm{z}{:}(\mathrm{A}\to\neg\mathrm{B}).\mathrm{z(a)(b)}$, [provided $\mathrm{z} \notin \mathrm{FV}_\lambda(\mathrm{a,b})$],
- $\pi_1(\mathrm{c{:}A\otimes B})$ $[\equiv @_1^\otimes(\mathrm{c{:}A\otimes B})]$ $:= \gamma \mathrm{z_0}{:}\neg\mathrm{A.c}(\lambda \mathrm{x_0{:}A}.\lambda \mathrm{y_0{:}B.z_0(x_0)})$,
- $\pi_2(\mathrm{c{:}A\otimes B})$ $[\equiv @_2^\otimes(\mathrm{c{:}A\otimes B})]$ $:= \gamma \mathrm{z_0}{:}\neg\mathrm{B.c}(\lambda \mathrm{x_0{:}A.z_0})$, [provided $\mathrm{z_0} \notin \mathrm{FV}_\lambda(\mathrm{c})$],

and, possibly, *mutatis mutandis*, as earlier,

- $\bigwedge_\otimes[\mathrm{x{:}A,y{:}B}].\mathrm{c}[\![\mathrm{x,y}]\!] \diamondsuit \mathrm{f}$ $[\equiv \bigwedge_\otimes(\mathrm{f{:}A\otimes B},[\mathrm{x{:}A}][\mathrm{y{:}B}].\mathrm{c}[\![\mathrm{x,y}]\!]{:}\mathrm{C})]$ $:= \mathrm{c}[\![\mathrm{x}{:=}\pi_1(\mathrm{f{:}A\otimes B})]\!][\![\mathrm{y}{:=}\pi_2(\mathrm{f{:}A\otimes B})]\!]$,

where we may also write $\pi_i(\mathrm{c})$ [or $\pi_i^\otimes(\mathrm{c})$, perhaps] and $\prec$ a, b $\succ$ resp., say, for the (intensional) "projections" $\pi_i(\mathrm{c{:}A\otimes B})$, [i := 1,2], and "pairs" $\prec$ a:A, b:B $\succ$, resp., if no confusions are likely.

(2) If we rely on [⊥,→] alone, there are analogous simulations of the Boolean ∨ with associated "intensional" operators $\gamma_\otimes$ [= η] and $\bigvee_\otimes$. Set, e.g., for $\mathrm{A} \oplus \mathrm{B} := \neg(\neg\mathrm{A} \otimes \neg\mathrm{B})$, [with $\otimes \equiv \otimes_\sqcap$ and $\oplus \equiv \oplus_\sqcap$], in analogy with the definitions of $\gamma_\natural$ [≡ **j**] and $\bigvee_\natural$ above:

- $\eta(\mathrm{x{:}\neg A,y{:}\neg B}).\mathrm{e}[\![\mathrm{x,y}]\!]$ $[\equiv \gamma_\otimes([\mathrm{x{:}\neg A}][\mathrm{y{:}\neg B}].\mathrm{e}[\![\mathrm{x,y}]\!]{:}\bot)]$ $:= \lambda \mathrm{z}{:}(\neg\mathrm{A}\otimes\neg\mathrm{B}).\mathrm{e}[\![\mathrm{x}{:=}\pi_1(\mathrm{z})]\!][\![\mathrm{y}{:=}\pi_2(\mathrm{z})]\!]$,
- $\bigvee_\otimes(\mathrm{z{:}\neg C}).\mathrm{f} \diamondsuit [\lambda \mathrm{x{:}A.e_1}[\![\mathrm{x,z}]\!],\lambda \mathrm{y{:}B.e_2}[\![\mathrm{y,z}]\!]]$ $[\equiv \bigvee_\otimes(\mathrm{f{:}A\oplus B},[\mathrm{x{:}A}][\mathrm{z{:}\neg C}].\mathrm{e_1}[\![\mathrm{x,z}]\!]{:}\bot,[\mathrm{y{:}B}][\mathrm{z{:}\neg C}].\mathrm{e_2}[\![\mathrm{y,z}]\!]{:}\bot]$ $:= \gamma \mathrm{z{:}\neg C.f}(\prec \lambda \mathrm{x{:}A.e_1}[\![\mathrm{x,z}]\!]{:}\neg\mathrm{A},\ \lambda \mathrm{y{:}B.e_2}[\![\mathrm{y,z}]\!]{:}\neg\mathrm{B} \succ)$.

(3) A completely similar situation obtains for the remaining (classically) definable pairs $[\otimes_\sqcup,\oplus_\sqcup]$, $[\otimes_\top,\oplus_\top]$, say, mentioned above. The simulation of the corresponding proof-operators in Boolean inferential terms is straightforward and is left as an *exercise*.[14]

Syntactic notions. The notions of a *subterm*, *free* and *bound* **U**- and *proof-variable* (*occurring*) *in* a proof-term are supposed to be defined, *mutatis mutandis*, as usually in typed λ-calculi. If a is a p-term, the corresponding sets of free/bound **U**- resp. p-variables are denoted by $\mathrm{FV}_u(\mathrm{a})$, $\mathrm{BV}_u(\mathrm{a})$, $\mathrm{FV}_\lambda(\mathrm{a})$, $\mathrm{BV}_\lambda(\mathrm{a})$ resp. A p-term a is **U**-*closed* if $\mathrm{FV}_u(\mathrm{a}) = \emptyset$, *p-closed* if $\mathrm{FV}_\lambda(\mathrm{a}) = \emptyset$ and *closed* if $\mathrm{FV}_u(\mathrm{a}) = \mathrm{FV}_\lambda(\mathrm{a}) = \emptyset$. The closed p-terms are called *Boolean proof-combinators*. One defines analogously the **U**-*open*, *p-open* and *open* p-terms. As above, the "window brackets" $[\![\ldots]\!]$ are metalinguistic artifices used to show the "internal" structure of a proof-term. E.g., "$\mathrm{c}[\![\mathrm{x}]\!]$" indicates the fact that the p-variable x has (possibly void) free occurrences in $\mathrm{c}[\![\mathrm{x}]\!]$. The substitution operators in proof-terms are shown by "$\mathrm{c}[\![\mathrm{x}{:=}\mathrm{a}]\!]$", "$\mathrm{c}[\![\mathrm{u}{:=}\mathbf{t}]\!]$", resp. They are supposed to be defined in the expected way. Throughout in what follows, we assume tacitly an appropriate notion of α-*conversion* (i.e., systematic relettering) $\equiv_\alpha$ for the *bound* **U**- and p-variables of a p-term and identify the resulting proof-term congruence relation with the *syntactic identity* ≡.

[14] Certainly, within [⊥,→,(∀)]-type-structures, many other intensional variants – more involved, although less interesting – for [⊗,⊕] are possible: in each case, the associated proof-operators would share intended stratification behaviors, whereas the equational behavior induced by the relevant $\lambda\gamma$-postulates should be rather *weak*.

Syntactic proof-environments. It is useful to have some notation for syntactic proof-environments. The notion of a *syntactic proof-environment* (or *applicative context*, *p-env*, for short) can be introduced intuitively.[15]Let $\bullet$ be an arbitrary symbol, distinct from those occurring already in the primitive proof-syntax. If $a[\![x]\!]$ is a proof-term containing a *single occurrence* of a *free* proof-variable x, then the proof-environment $a[\![\bullet]\!] \equiv a[\![x{:=}\bullet]\!]$ is the word obtained from $a[\![x]\!]$, by *substituting* $\bullet$ *for the actual occurrence of* x *in* a. The dummy "$\bullet$" plays here the rôle of a "hole-marker" in a. Formally, one defines p-env's by induction, in the obvious way.

Notation (*Syntactic proof-environments*).

> We let $\varphi\ll\bullet\gg$ (possibly decorated) range over p-env's. If a is a p-term and $\varphi\ll\bullet\gg$ is a p-env then $\varphi\ll a\gg \equiv \varphi\ll\bullet{:=}a\gg$. Here $\varphi\ll a\gg$ is the p-term obtained by substituting a for the dummy $\bullet$ in $\varphi\ll\bullet\gg$, *provided* a does not contain free **U**- and p-variables that are bound in $\varphi\ll\bullet\gg$ (i.e., **U**- and p-variables occurring within the scope of an abstractor, "hidden" in $\varphi\ll\bullet\gg$).

This allows the use of "windows" *and* substitutions *inside* p-envs. *Example*. If $\varphi\ll\bullet\gg$ is a p-env, then we can write $\varphi\ll x(\gamma y{:}\neg A.c[\![x,y]\!]) \gg$ and $\varphi\ll c[\![x{:=}z]\!][\![y{:=}z]\!] \gg$, without ambiguity, assuming that the *free* p-variables x, z are not captured by a variable binder in $\varphi\ll\bullet\gg$ (and similarly about the free **U**- and p-variables of c). In such cases we are still allowed to "bind" x or z *ouside* the p-env $\varphi\ll\bullet\gg$, by, e.g., $\lambda x{:}\neg A.\varphi\ll x(\gamma y{:}\neg A.c[\![x,y]\!]) \gg$ or $\gamma z{:}\neg A.\varphi\ll c[\![x{:=}z]\!][\![y{:=}z]\!] \gg$, etc.

Proof-operators/proof-terms in Minimalkalkül and the Heyting logic. The positive sumptors and the (general) positive selectors – as well as the standard "application forms" – listed above are, in fact, *Minimalkalkül* proof-operators and, therefore, they make sense intuitionistically, too (that is: they are proof-operators of the first-order Heyting logic **HQ**).

As regards the [∨,∃]-proof-operators, the proof-theory of **MQ** – essentially, the first-order fragment of Martin-Löf's [84] constructive type theory – and **HQ** would allow as meaningful only the *positive uses* of the Boolean negative [∨,∃]-proof-operators. In a Boolean setting, this limitation admits of a formal – abstract – characterization.

We consider first **MQ**-proof-structures. Relative to proof-languages based on a [⊥,→,∧,∨,∀,∃]-type-structure, the "positive" ("minimal") contents of the Boolean negative sumptors (i.e., the "classical injections") **j**, **J** resp., and the negative selectors $\bigvee_\natural$ and $\bigvee_\cup$ resp., can be recorded by the following

Definition (*"Minimal"* [∨,∃]-*proof-operators: instantiation*).

(1) *"Minimal"* [∨,∃]-*injections*:

- $\mathbf{j}_1[\![A,B]\!](a{:}A) := \mathbf{j}(x{:}\neg A,y{:}\neg B).x(a)$, $[x, y \notin FV_\lambda(a)]$,
- $\mathbf{j}_2[\![A,B]\!](b{:}B) := \mathbf{j}(x{:}\neg A,y{:}\neg B).y(b)$, $[x, y \notin FV_\lambda(b)]$,
- $[\mathbf{t},a{:}A[\![\mathbf{t}]\!]] := \mathbf{J}(z{:}\neg A[\![\mathbf{t}]\!]).z(a)$, $[z \notin FV_\lambda(a)]$.

(2) *"Minimal"* [∨,∃]-*selectors*:

- $\sqcup(f,[x{:}A].c_1[\![x]\!]{:}C,[y{:}B].c_2[\![y]\!]{:}C) := \bigvee_\natural(z{:}\neg C).f \diamondsuit [\lambda x{:}A.z(c_1[\![x]\!]),\lambda y{:}B.z(c_2[\![y]\!])]$,
  $[z \notin FV_\lambda(c_1[\![x]\!],c_2[\![y]\!])]$,
- $\amalg(f,[u{:}\mathbf{U}][x{:}A[\![u]\!]].c[\![u,x]\!]{:}C) := \bigvee_\cup(z{:}\neg C).f \diamondsuit [!u.\lambda x{:}A[\![u]\!].z(c[\![u,x]\!])]$,
  $[z \notin FV_\lambda(c[\![u,x]\!]),u \notin FV_u(C)]$.

If confusions are unlikely, we may leave out [some/all] type-parameters, writing, e.g., also $\mathbf{j}_i[\![A_1,A_2]\!](a_i)$ for $\mathbf{j}_i[\![A_1,A_2]\!](a_i{:}A_i)$ $[i := 1,2]$, $[\mathbf{t},a]$ for $[\mathbf{t},a{:}A[\![\mathbf{t}]\!]]$ and $\sqcup(f,[x{:}A].c_1[\![x]\!],[y{:}B].c_2[\![y]\!])$, resp. $\amalg(f,[u{:}\mathbf{U}][x{:}A[\![u]\!]].c[\![u,x]\!])$

[15]Like its close relative, used currently in theoretical computer science, this ingredient reflects a *local notational policy* and – as far as the present work is concerned – should not be confused with the concept of a *proof-context* [= *virtual* proof-set, "assumption-set"].

for $\sqcup(\mathrm{f},[\mathrm{x{:}A}].\mathrm{c}_1\llbracket \mathrm{x}\rrbracket{:}\mathrm{C},[\mathrm{y{:}B}].\mathrm{c}_2\llbracket \mathrm{y}\rrbracket{:}\mathrm{C})$ resp. $\amalg(\mathrm{f},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket \mathrm{u}\rrbracket].\mathrm{c}\llbracket \mathrm{u,x}\rrbracket{:}\mathrm{C})$. As a notational alternative, we shall use oft the operator-form $\mathbf{J}_m(\mathrm{a{:}A}\llbracket \mathbf{t}\rrbracket)$ for $[\mathbf{t},\mathrm{a{:}A}\llbracket \mathbf{t}\rrbracket]$ (i.e., for the "minimal" ∃-injection).

Remark (*"Minimal"-*[∨,∃]*-proofs and "negative" assumptions*). From the above it should be obvious that the "minimal" instances of the Boolean negative [∨,∃]-proof-operators obey a general principle that could be expressed by the recommendation: "do *never* use *negative* information in [∨,∃]-proofs" (or just: "take positive contents only"). In other words, the explicit "negative" assumptions, functioning as *local* context-parameters in a Boolean proof-operator are *not* used actually in the **MQ**-version of the operator. For instance, if the proof-variable z is not in $\mathrm{FV}_\lambda(\mathrm{c}_1\llbracket \mathrm{x}\rrbracket,\mathrm{c}_2\llbracket \mathrm{y}\rrbracket)$, the negative **MQ**-selector $\sqcup(\mathrm{f},[\mathrm{x{:}A}].\mathrm{c}_1\llbracket \mathrm{x}\rrbracket{:}\mathrm{C},[\mathrm{y{:}B}].\mathrm{c}_2\llbracket \mathrm{y}\rrbracket{:}\mathrm{C}) : \mathrm{C}$ represents a *special* (case of the Boolean) operator-form

$$\bigvee_\natural(\mathrm{f{:}A{\vee}B},[\mathrm{x{:}A}][\mathrm{z{:}\neg C}].\mathrm{z}(\mathrm{c}_1\llbracket \mathrm{x}\rrbracket){:}\bot,[\mathrm{y{:}B}][\mathrm{z{:}\neg C}].\mathrm{z}(\mathrm{c}_2\llbracket \mathrm{y}\rrbracket){:}\bot) : \mathrm{C},$$

(term-form: $\bigvee_\natural(\mathrm{z{:}\neg C}).\mathrm{f} \diamondsuit [\lambda \mathrm{x{:}A}.\mathrm{z}(\mathrm{c}_1\llbracket \mathrm{x}\rrbracket), \lambda \mathrm{y{:}B}.\mathrm{z}(\mathrm{c}_2\llbracket \mathrm{y}\rrbracket)]$).

In the case of the "minimal" ∨-injections, the "positive only" policy amounts to a splitting (of the action) of the Boolean injection $\gamma_\natural$ [≡ **j**] (into separate actions):

$$\gamma_\natural([\mathrm{x{:}\neg A}][\mathrm{y{:}\neg B}].\mathrm{x}(\mathrm{a}){:}\bot) \equiv \mathbf{j}(\mathrm{x{:}\neg A,y{:}\neg B}).\mathrm{x}(\mathrm{a}) \equiv \mathbf{j}_1\llbracket \mathrm{A,B}\rrbracket(\mathrm{a{:}A}),$$
$$\gamma_\natural([\mathrm{x{:}\neg A}][\mathrm{y{:}\neg B}].\mathrm{y}(\mathrm{b}){:}\bot) \equiv \mathbf{j}(\mathrm{x{:}\neg A,y{:}\neg B}).\mathrm{y}(\mathrm{b}) \equiv \mathbf{j}_2\llbracket \mathrm{A,B}\rrbracket(\mathrm{b{:}B}),$$

where x, y are not in $\mathrm{FV}_\lambda(\mathrm{a})$ resp. $\mathrm{FV}_\lambda(\mathrm{b})$. A *side-effect* of the restriction is in the fact that the resulting (*special*) injections $\mathbf{j}_i\llbracket \mathrm{A,B}\rrbracket$, [i := 1,2], become *type-parametric* in a sense which is *not intended* in the general Boolean case (!), where the actual dependencies must be expressed in terms of proof-[term]s and not in terms of types ["propositions"/formulas]. Clearly, this type of restriction applies uniformly to both the ∨- *and* the ∃-proof-operators.

The Heyting logic **HQ** extends the "positive limiting" policy to the action of the (negative) *inferential proof-operators* (i.e., in the present setting, to the γ-abstractions), with, also, an unintended *type-parametrization* as a side-effect.[16]

Without [∨,∃]-(proof)-primitives, we can *simulate*, in a Boolean setting, the "positive"/"minimal" proof-operators along the general pattern displayed above. Specifically, relative to the [⊥,→,∧,∀]-type-structure, in view of the standard – Boolean – definitions of ∨, ∃ and the associated proof-operators **j**, $\bigvee_\natural$, resp. **J**, $\bigvee_\cup$, one has the following identities (turned into definitional stipulations):

Definition (*"Minimal"* [∨,∃]*-proof-operators*: *Boolean simulation*).

(1) *"Minimal"* [∨]*-proof-operators*:

- $\mathbf{j}_1\llbracket \mathrm{A,B}\rrbracket(\mathrm{a{:}A}) := \lambda \mathrm{z}{:}(\neg \mathrm{A}\wedge\neg \mathrm{B}).\mathbf{p}_1(\mathrm{z})(\mathrm{a})$, $[\mathrm{z} \notin \mathrm{FV}_\lambda(\mathrm{a})]$,
- $\mathbf{j}_2\llbracket \mathrm{A,B}\rrbracket(\mathrm{b{:}B}) := \lambda \mathrm{z}{:}(\neg \mathrm{A}\wedge\neg \mathrm{B}).\mathbf{p}_2(\mathrm{z})(\mathrm{b})$, $[\mathrm{z} \notin \mathrm{FV}_\lambda(\mathrm{b})]$,
- $\sqcup(\mathrm{f},[\mathrm{x{:}A}].\mathrm{c}_1\llbracket \mathrm{x}\rrbracket{:}\mathrm{C},[\mathrm{y{:}B}].\mathrm{c}_2\llbracket \mathrm{y}\rrbracket{:}\mathrm{C}) := \gamma \mathrm{z{:}\neg C}.(\mathrm{f})\langle\lambda \mathrm{x{:}A}.\mathrm{z}(\mathrm{c}_1\llbracket \mathrm{x}\rrbracket){:}\neg \mathrm{A},\lambda \mathrm{y{:}B}.\mathrm{z}(\mathrm{c}_2\llbracket \mathrm{y}\rrbracket){:}\neg \mathrm{B}\rangle$,
  $[\mathrm{z} \notin \mathrm{FV}_\lambda(\mathrm{c}_1\llbracket \mathrm{x}\rrbracket,\mathrm{c}_2\llbracket \mathrm{y}\rrbracket)]$,

(2) *"Minimal"* [∃]*-proof-operators*:

- $[\mathbf{t},\mathrm{a{:}A}\llbracket \mathbf{t}\rrbracket]$ $[\equiv \mathbf{J}_m(\mathrm{a{:}A}\llbracket \mathbf{t}\rrbracket)]$ $:= \lambda \mathrm{z}{:}(\forall \mathrm{u}.\neg \mathrm{A}\llbracket \mathrm{u}\rrbracket).\mathrm{z}[\mathbf{t}](\mathrm{a})$, $[\mathrm{z} \notin \mathrm{FV}_\lambda(\mathrm{a})]$,
- $\amalg(\mathrm{f},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket \mathrm{u}\rrbracket].\mathrm{c}\llbracket \mathrm{u,x}\rrbracket{:}\mathrm{C}) := \gamma \mathrm{z{:}\neg C}.\mathrm{f}(!\mathrm{u}.\lambda \mathrm{x{:}A}\llbracket \mathrm{u}\rrbracket.\mathrm{z}(\mathrm{c}\llbracket \mathrm{u,x}\rrbracket))$, $[\mathrm{z} \notin \mathrm{FV}_\lambda(\mathrm{c}\llbracket \mathrm{u,x}\rrbracket),\mathrm{u} \notin \mathrm{FV}_u(\mathrm{C})]$.

It is easy to establish the fact that the "minimal" injections and selectors thereby defined do actually correspond to the intended *Minimalkalkül* proof-operators, i.e., to the [∨,∃]-proof-operators of the Johansson **MQ**-logic, and therefore also to *intuitionistic* [*Brouwerian*] proof-operators. Notably, these operators can be shown to satisfy the expected extensionality assumptions.

As suggested in the above, there are analogous – though "intensional" – simulations of the **MQ** [∧,∨]-proof-operators in terms of the Boolean proof-operators associated to [appropriate definitions of] ⊗ and ⊕, with,

[16] For details, see the discussion of the ω-operator(s) appearing below.

e.g., $A\otimes B \equiv A\otimes_{\sqcap}B \equiv \neg(A\to\neg B)$, $A\oplus B \equiv A\oplus_{\sqcap}B \equiv \neg(\neg A\otimes_{\sqcap}\neg B) \equiv \neg\neg(\neg A\to\neg\neg B)$, etc. It is immediate that the proof-operators that can be associated to the definable (Boolean) pairs $[\otimes,\oplus]$, listed earlier, specialize ("positively") to analogous – although (equationally) distinct – "intensional" **MQ**-operators [*exercise*].

Beyond Minimalkalkül 1: "Clavian" instances of the Boolean $[\vee,\exists]$*-proof-operators*. The Boolean "injections" **j**, **J**, resp. and the Boolean $[\vee,\exists]$-selectors admit of slightly more general instantiations than in the case of the *Minimalkalkül* notions. These are "Clavian" variants of the Boolean $[\vee,\exists]$-proof-operators. Indeed, the corresponding rule-forms require "Clavian" hypotheses, i.e., proof-patterns $\Gamma[x{:}\neg A] \vdash a\llbracket x\rrbracket : A$ (as in the premise of the so-called "Rule of Clavius": $\neg A \Vdash A \Rightarrow \Vdash A$).

Definition (*"Clavian"* $[\vee,\exists]$*-proof-operators*: *instantiation*).

(1) *"Clavian"* $[\vee]$*-proof-operators*:

- $\mathbf{j}^1(x{:}\neg A,\ y{:}\neg B).a\llbracket x,y\rrbracket \quad := \mathbf{j}(x{:}\neg A,y{:}\neg B).x(a\llbracket x,y\rrbracket)$,
- $\mathbf{j}^2(x{:}\neg A,y{:}\neg B).b\llbracket x,y\rrbracket \quad := \mathbf{j}(x{:}\neg A,y{:}\neg B).y(b\llbracket x,y\rrbracket)$,
- $\sqcup_d(z{:}\neg C).f \diamondsuit [\lambda x{:}A.c_1\llbracket x,z\rrbracket{:}C,\ \lambda y{:}B.c_2\llbracket y,z\rrbracket{:}C] \quad := \bigvee_{\natural}(z{:}\neg C).f \diamondsuit [\lambda x{:}A.z(c_1\llbracket x,z\rrbracket),\ \lambda y{:}B.z(c_2\llbracket y,z\rrbracket)]$,

(2) *"Clavian"* $[\exists]$*-proof-operators*:

- $\mathbf{J}_d(x{:}\neg A\llbracket u{:=}\mathbf{t}\rrbracket).a\llbracket x\rrbracket \quad := \mathbf{J}(x{:}\neg A\llbracket u{:=}\mathbf{t}\rrbracket).x(a\llbracket x\rrbracket)$,
- $\amalg_d(z{:}\neg C).f \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.c\llbracket u,x,z\rrbracket{:}C] \quad := \bigvee_{\cup}(z{:}\neg C).f \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.z(c\llbracket u,x,z\rrbracket)]$, $[u \notin FV_u(C)]$.

In a Boolean setting, the "minimal" injections and the $[\vee,\exists]_m$-selectors are *special cases* of the "Clavian" injections and the $[\vee,\exists]_d$-selectors, resp., viz.

(i) "minimal" *vs* "Clavian" injections:

- $\mathbf{j}_1\llbracket A,B\rrbracket(a) \quad \equiv \mathbf{j}^1(x{:}\neg A,y{:}\neg B).a \quad [\equiv \mathbf{j}(x{:}\neg A,y{:}\neg B).x(a)]$, $[x,\ y \notin FV_\lambda(a)]$,
- $\mathbf{j}_2\llbracket A,B\rrbracket(b) \quad \equiv \mathbf{j}^2(x{:}\neg A,y{:}\neg B).b \quad [\equiv \mathbf{j}(x{:}\neg A,y{:}\neg B).y(b)]$, $[x,\ y \notin FV_\lambda(b)]$,
- $[\mathbf{t},a{:}A\llbracket u{:=}\mathbf{t}\rrbracket] \equiv \mathbf{J}_m(a{:}A\llbracket u{:=}\mathbf{t}\rrbracket) \quad \equiv \mathbf{J}_d(x{:}\neg A\llbracket u{:=}\mathbf{t}\rrbracket).a \quad [\equiv \mathbf{J}(x{:}\neg A\llbracket u{:=}\mathbf{t}\rrbracket).x(a)]$, $[x \notin FV_\lambda(a)]$,

(ii) "minimal" *vs* "Clavian" (negative) selectors:

- $\sqcup(f,[x{:}A].c_1\llbracket x\rrbracket{:}C,[y{:}B].c_2\llbracket y\rrbracket{:}C) \quad \equiv \sqcup_d(z{:}\neg C).f \diamondsuit [\lambda x{:}A.c_1\llbracket x\rrbracket{:}C,\lambda y{:}B.c_2\llbracket y\rrbracket{:}C]$,
  $[z \notin FV_\lambda(c_1\llbracket x\rrbracket,c_2\llbracket y\rrbracket)]$,
- $\amalg(f,[u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket].c\llbracket u,x\rrbracket{:}C) \quad \equiv \amalg_d(z{:}\neg C).f \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.c\llbracket u,x\rrbracket{:}C]$,
  $[z \notin FV_\lambda(c\llbracket u,x\rrbracket),\ u \notin FV_u(C)]$.

As expected, one can *simulate* the "Clavian" proof-operators too, in a Boolean setting (without $[\vee,\exists]$-primitives, relative to the $[\bot,\to,\wedge,\forall]$-type-structure), *via* the general pattern above.

Definition (*"Clavian"* $[\vee,\exists]$*-proof-operators*: *Boolean simulation*)

(1) *"Clavian"* $[\vee]$*-proof-operators*:

- $\mathbf{j}^1(x{:}\neg A,y{:}\neg B).a\llbracket x,y\rrbracket \ := \lambda z{:}(\neg A\wedge\neg B).\mathbf{p}_1(z)(a\llbracket x{:=}\mathbf{p}_1(z)\rrbracket)$, $[z \notin FV_\lambda(a\llbracket x,y\rrbracket)]$,
- $\mathbf{j}^2(x{:}\neg A,y{:}\neg B).b\llbracket x,y\rrbracket \ := \lambda z{:}(\neg A\wedge\neg B).\mathbf{p}_2(z)(b\llbracket y{:=}\mathbf{p}_2(z)\rrbracket)$, $[z \notin FV_\lambda(b\llbracket x,y\rrbracket)]$,
- $\sqcup_d(z{:}\neg C).f \diamondsuit [\lambda x{:}A.c_1\llbracket x,z\rrbracket{:}C,\lambda y{:}B.c_2\llbracket y,z\rrbracket{:}C] \ := \gamma z{:}\neg C.(f)<\lambda x{:}A.z(c_1\llbracket x,z\rrbracket){:}\neg A,\lambda y{:}B.z(c_2\llbracket y,z\rrbracket){:}\neg B>$,

(2) *"Clavian"* $[\exists]$*-proof-operators*:

- $\mathbf{J}_d(x{:}\neg A\llbracket\mathbf{t}\rrbracket).a\llbracket x\rrbracket \quad := \lambda z{:}(\forall u.\neg A\llbracket u\rrbracket).z[\mathbf{t}](a\llbracket x{:=}z[\mathbf{t}]\rrbracket)$, $[z \notin FV_\lambda(a)]$,
- $\amalg_d(z{:}\neg C).f \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket].c\llbracket u,x,z\rrbracket{:}C) \quad := \gamma z{:}\neg C.f(!u.\lambda x{:}A\llbracket u\rrbracket.z(c\llbracket u,x,z\rrbracket))$, $[u \notin FV_u(C)]$.

The "Clavian" proof-operators so defined correspond to proof-operators of the logic of "complete refutability" (here: **DQ**) of H. B. Curry [52,63], also known as a "logic of strict negation".[17]In contrast with the "minimal"

[17]The appellation "Clavian" – frequent in professional logic jargon, although hardly documented in logic text-books – alludes to Christoph Klau SJ [*Lat. Clavius*] (1537-1612), a Jesuit mathematician and astronomer, credited oft – erroneously – by a hasty – first Jesuit, later common – tradition, with the "discovery" of the *consequentia mirabilis* $[\neg A \to A \to A]$. Historically,

[$\vee$,$\exists$]-notions, the "Clavian" notions are essentially *non-Brouwerian* (so, they are *non-Boolean*, rather than *non-classical*). Indeed, they yield *tertium non datur* by mere (type-) instantiation, although their inferential properties alone are *not* sufficient in order to express the most general form of, say, *reductio ad absurdum* (and, in fact, not even the most general form of *ex falso quodlibet*).

Beyond Minimalkalkül 2: the Heyting logic and ex falso quodlibet. As is well-known, the Boolean tautology $\bot \to$ A [*ex falso quodlibet*] is valid in the Heyting logic **HQ** and can be also used (e.g., in axiomatics) in order to distinguish **HQ** from **MQ** (in fact, one has: **HQ** = **MQ** + [$\bot \to$ A]). The following definitions simulate the specific intuitionistic proof-operator(s) involved in *ex falso quodlibet* in Boolean terms, relative to a [$\bot$,$\to$,($\wedge$,$\vee$,$\forall$,$\exists$)]-type-structure,

Definition (*Ex falso quodlibet and the* $\omega[\![A]\!]$-*family*, $\Vdash$ A :: **H**).

For all formulas A in [$\bot$,$\to$,($\wedge$,$\vee$,$\forall$,$\exists$)], set

- $\omega_A(e{:}\bot)$ [$\equiv \omega(e{:}\bot){:}A$] := $\gamma x{:}\neg A.e$ [$\equiv \gamma_\vdash([x{:}\neg A].e{:}\bot)$], **where** $x \notin FV_\lambda(e)$,
- $\omega[\![A]\!]$:= $\lambda x{:}\bot.\omega_A(x)$ [$\equiv \lambda x{:}\bot.\gamma y{:}\neg A.x$].

For any formula A in [$\bot$,$\to$,($\wedge$,$\vee$,$\forall$,$\exists$)], we write also oft $\omega_A(e)$ for $\omega_A(e{:}\bot)$, if the context is clear. In the above, $\omega[\![A]\!]$ (for A in [$\bot$,$\to$,($\wedge$,$\vee$,$\forall$,$\exists$)]) stands for a family of (*genuine*) *Heyting proof-combinators*.

So, the (Heyting) proof-operator $\omega_A(e{:}\bot)$ [$= \gamma_\vdash([x{:}\neg A].e{:}\bot)$, where $x \notin FV_\lambda(e)$] takes the "positive contents" of the Boolean negative (inferential) proof-operator $\gamma_\vdash(x{:}\neg A.e{:}\bot)$ in a sense very much similar to the way the **MQ** [$\vee$,$\exists$]-proof-operators were seen to "take (only) positive contents" from the analogous Boolean [$\vee$,$\exists$]-proof-operators. Here, indeed, the "negative" assumption (= local context-parameter) [z : $\neg$A] is never used, since the proof-situation depicted formally by

$$\gamma_\vdash([x{:}\neg A].e{:}\bot) : A, \text{ provided } x \notin FV_\lambda(e)$$

means that the proof e (of $\bot$) does not actually "depend on" the assumption [z : $\neg$A] (although it may possibly "depend on" global context-parameters, ignored in the above). In this case, the redundant ("negative") assumption can be safely taken care of by a different kind of rule (viz., by a proof-context rule, i.e., in the present setting, by the "cut"-rule $< \$K >$).

Remark (*"Finitary"/local γ-abstractions in the Heyting logic*). As suggested earlier, for every formula A in [$\bot$,$\to$,$\wedge$,$\vee$,$\forall$,$\exists$] such that A is "decidable" intuitionistically, i.e., whenever $\neg$A $\vee$ A *has an intuitionistic proof f*, one can simulate [in classical logic] an intuitionistically "correct" *local* γ-abstractor γ_f, *depending on f* [in fact, on *the Boolean image of f*, as obtained *via* the definitional embedding implicit in the above]. A possible "definition" could be, for instance,

$$\gamma_f x{:}\neg A.e[\![x]\!] := \sqcup(f{:}\neg A \vee A,[x{:}\neg A].\omega_A(e[\![x]\!]){:}A,[y{:}A].y{:}A).$$

the implied attribution is at least amusing, since the only references [two] in Clavius' rather verbose *Opera mathematica* [five tomes, *in folio*] to what we now use to call the "Law of Clavius" are there mainly in order to fight ignorance, and say, essentially, that sample uses of this "law" – *mirabile argumentandi modus*, indeed, to his mind – could have been already encountered in the Euclidean *Elements* [*etiam usus est Euclides* (IX.12)], so that, implicitly, no one, among the *recentiores* – even as famous as Gerolamo Cardano (1501-1576), by the time Clavius was writing –, could possibly claim he got it "first". For the Pavian, this figure of proof was not less than *res admirabilior quae inventa sit ab urbe condito..., longe majus Chrysippaeo syllogismo...* (*De proportionibus*, [Basle 1570]; cf. [Cardano 1663] 4, 579), and he seemed very proud of having found it, on his own [after all, this is true]. Better documented, the Jesuit noticed that he wasn't the first one to have used it in the history of mathematics or – *pace* Chrysippus – in the *history of proving* [for Stoic antecedents, see, e.g., Sextus Emp. *Adv. math.*, VIII, 281-2, 466 *sq*.]. On the other hand, once we decide to pay attention to the incidence of the *relevant theoretical detail* in [the history of] logic, the adjective "Saccherian", for instance – alluding, this time properly, also to a Jesuit scholar (!) – becomes, in this context, a better substitute. Since it is, anyway, too late to change the logician's traditional onomastics, we leave things as they are – and as they were some 400 years ago, inasfar *Pater Clavius* is concerned –, extending, however, the use of the epi-theoretic qualifier "Clavian" such as to cover the full range of proof-phenomena – operations, relations, properties, etc. – induced by the presence of genuinely "Clavian proof-patterns" $\Gamma[x{:}\neg A] \vdash a[\![x]\!] : A$.

In a genuine Boolean setting, this is uninteresting, since the “intuitionistic” proof-terms $\omega_A(e[\![x{:}\neg A]\!])$ are just special γ-abstractions $[\gamma y{:}\neg A.e[\![x]\!]$, with y not free in $e[\![x]\!]$, i.e., with x and y being distinct proof-variables of type $\neg A]$.

Chapter III

Calculi of $\lambda\gamma$-conversion

In this section, we introduce a typed λ-calculus $\lambda\gamma_0\mathbf{CQ}$, based on the type-structure $[\bot,\rightarrow,\wedge,\forall]$. As expected, this is an *equational theory* $\lambda\gamma_0\mathbf{CQ} = < \vdash_0[\mathbf{CQ}], =(\lambda\gamma_0\mathbf{CQ}) >$, consisting of

- *a stratification* $\vdash_0[\mathbf{CQ}]$ for the proof-terms of $\mathbf{CQ}$; ($\vdash_0[\mathbf{CQ}]$ defines the *classical consequence relation* for $[\bot,\rightarrow,\wedge,\forall]$-languages or *the concept of a classical proof* relative to $[\bot,\rightarrow,\wedge,\forall]$), and
- a concept of *proof-equality* $=(\lambda\gamma_0\mathbf{CQ})$, defined on the proof-terms generated by $\vdash_0[\mathbf{CQ}]$.

Type-theoretically, a *stratification* is a partial map from p-terms to formulas. Its graph can be viewed as a set of *proof-statements* of the form $\Gamma \vdash_0$ a : A (read: "a proves A in the proof-context Γ"). $\vdash_0[\mathbf{CQ}]$ is given by (*stratification*) *rules*: these consist of *proof-context rules* and (*proper*) *type-assignment rules* (the so-called "derivation rules", describing the structure of the Boolean proof-operators). For the sake of simplicity, we consider that the proof-contexts are (assumption-) *sets*.[18]Without loss of generality, the (p-term-) stratification $\vdash_0[\mathbf{CQ}]$ can be identified with a set of proof-terms (the "Boolean proof-terms", i.e., the proof-terms that are "stratifiable" relative to $\vdash_0[\mathbf{CQ}]$).

The equational postulates – in common parlance: "rules" – of the proof-calculus $\lambda\gamma_0\mathbf{CQ}$ have the general form $\Gamma \vdash a_1 = a_2$ [: A], *provided* $\Gamma \vdash a_1, a_2$: A. For convenience, we state the stratification conditions explicitly. In particular, the proof-term equality of $\lambda\gamma_0\mathbf{CQ}$ turns out to be *finitely axiomatizable*, in the *schematic* sense. Of course, the properties of $\vdash_0[\mathbf{CQ}]$ do not depend on $=(\lambda\gamma_0\mathbf{CQ})$.

Subsequently, $\lambda\gamma_0\mathbf{CQ}$ is extended to a definitionally equivalent structure (equational theory) $\lambda\gamma_\&\mathbf{CQ} = < \vdash_\&[\mathbf{CQ}], =(\lambda\gamma_\&\mathbf{CQ}) >$. The latter – based on $[\bot,\rightarrow,\wedge,\forall]$, too – can be viewed as the "official" *proof-theory of first-order classical logic*.[19]

Definition ($\vdash_0[\mathbf{CQ}]$ *and* $\lambda\gamma_0\mathbf{CQ}$).

(1) *First-order Boolean proof-term stratification*.

(11) *Proof-context rules*, as above:

1.1.1 "structural": $< I >$, $< K >$, $< K_u >$, and
1.1.2 "cuts": $< \$ >$, $< \$_{[u]} >$ (possibly: $< \$K >$, $< \$_u K >$).

(12) *The Boolean "type-assignment"*.

1.2.1 Inferential rules.

$(\rightarrow i\lambda)$ $\quad \Gamma[\text{x:A}] \vdash \text{b}[\![\text{x}]\!] : \text{B} \quad \Rightarrow \Gamma \vdash \lambda\text{x:A.b}[\![\text{x}]\!] : \text{A} \rightarrow \text{B}$,
$(\rightarrow e@_{\vdash})$ $\quad \Gamma_1 \vdash \text{f} : \text{A} \rightarrow \text{B}, \Gamma_2 \vdash \text{a} : \text{A} \quad \Rightarrow \Gamma_1\Gamma_2 \vdash \text{f(a)} : \text{B}$,
$(\rightarrow i\gamma)_0$ $\quad \Gamma[\text{x:}\neg\text{A}] \vdash \text{e}[\![\text{x}]\!] : \bot \quad \Rightarrow \Gamma \vdash \gamma\text{x:}\neg\text{A.e}[\![\text{x}]\!] : \text{A}$, **where** A is a "prime" formula.

1.2.2 Algebraic rules.

$(\wedge i)$ $\quad \Gamma_1 \vdash \text{a} : \text{A}, \Gamma_2 \vdash \text{b} : \text{B} \Rightarrow \Gamma_1\Gamma_2 \vdash \text{<a:A,b:B>} : \text{A} \wedge \text{B}$,
$(\wedge e@_1^\natural)$ $\quad \Gamma \vdash \text{f} : \text{A} \wedge \text{B} \quad \Rightarrow \Gamma \vdash \mathbf{p}_1(\text{f:A}\wedge\text{B}) : \text{A}$,
$(\wedge e@_2^\natural)$ $\quad \Gamma \vdash \text{f} : \text{A} \wedge \text{B} \quad \Rightarrow \Gamma \vdash \mathbf{p}_2(\text{f:A}\wedge\text{B}) : \text{B}$.

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[18]The graph of the *partial* map $\vdash_0$: p-terms $\hookrightarrow$ formulas, $\Im(\vdash_0)$ say, is thus supposed to be generated effectively from an "atomic" type-assignment ϱ : Λ-atoms $\longrightarrow$ formulas. So, each point of $\Im(\vdash_0)$ is *a pair* (a : A), where $\text{A} \equiv \text{A}[\![\text{u}_1\text{:}\mathbf{U},\ldots,\text{u}_m\text{:}\mathbf{U}]\!]$ and $\text{a} \equiv \text{a}[\![\text{u}_1\text{:}\mathbf{U},\ldots,\text{u}_m\text{:}\mathbf{U},\text{x}_1\text{:A}_1,\ldots,\text{x}_n\text{:A}_n]\!]$, with parameters (i.e., $\mathbf{U}$- and Λ-atoms), occurring *free* (in A, and a) only as shown and where the parameter-stratification is determined by a *finite* segment of (the graph of) ϱ, viz. by a proof-context $\Gamma \equiv [\text{u}_1\text{:}\mathbf{U}]\ldots[\text{u}_m\text{:}\mathbf{U}] \smile [\text{x}_1\text{:A}_1]\ldots[\text{x}_n\text{:A}_n]$.

[19]Here, the *technical* concept of a *proof-theory* (*$\lambda\gamma$-theory*) functions as an *explanatum* for an *intuitive* notion/practice.

1.2.3 Generic [first-order] rules.

$(\forall i)$	$\Gamma[\mathrm{u}{:}\mathbf{U}] \vdash \mathrm{a}\llbracket \mathrm{u}\rrbracket : \mathrm{A}\llbracket \mathrm{u}\rrbracket$	$\Rightarrow \Gamma \vdash !\mathrm{u}.\mathrm{a}\llbracket \mathrm{u}\rrbracket : \forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket$,
$(\forall e@_{\cup})$	$\Gamma \vdash \mathrm{f} : \forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket,\ \Gamma_u \Vdash \mathbf{t} :: \mathbf{U}$	$\Rightarrow \Gamma_u\Gamma \vdash \mathrm{f}[\mathbf{t}] : \mathrm{A}\llbracket \mathrm{u}{:=}\mathbf{t}\rrbracket,\ [\mathrm{u} \notin \mathrm{FV}_u(\mathrm{f})]$.

(2) *First-order Boolean proof-equality* [$\lambda\gamma_0$**CQ**-*equality*].

2.1 Functional rules [$= \lambda\pi!$-*equality*].

$[\beta \to \lambda]$	$\Gamma \vdash (\lambda \mathrm{x}{:}\mathrm{A}.\mathrm{b}\llbracket \mathrm{x}\rrbracket)(\mathrm{a}) = \mathrm{b}\llbracket \mathrm{x}{:=}\mathrm{a}\rrbracket\ [: \mathrm{B}]$,	if $\Gamma \vdash \mathrm{a} : \mathrm{A}$, and $\Gamma[\mathrm{x}{:}\mathrm{A}] \vdash \mathrm{b}\llbracket \mathrm{x}\rrbracket : \mathrm{B}$,
$[\eta \to \lambda]$	$\Gamma \vdash \lambda \mathrm{x}{:}\mathrm{A}.\mathrm{f}(\mathrm{x}) = \mathrm{f}\ [: \mathrm{A} \to \mathrm{B}]$,	if $\Gamma \vdash \mathrm{f} : \mathrm{A} \to \mathrm{B}$, $[\mathrm{x} \notin \mathrm{FV}_\lambda(\mathrm{f})]$,
$[\beta\wedge_1]$	$\Gamma \vdash \mathbf{p}_1(<\mathrm{a}{:}\mathrm{A},\mathrm{b}{:}\mathrm{B}>) = \mathrm{a}\ [: \mathrm{A}]$,	if $\Gamma \vdash \mathrm{a} : \mathrm{A}$, and $\Gamma \vdash \mathrm{b} : \mathrm{B}$,
$[\beta\wedge_2]$	$\Gamma \vdash \mathbf{p}_2(<\mathrm{a}{:}\mathrm{A},\mathrm{b}{:}\mathrm{B}>) = \mathrm{b}\ [: \mathrm{B}]$,	if $\Gamma \vdash \mathrm{a} : \mathrm{A}$, and $\Gamma \vdash \mathrm{b} : \mathrm{B}$,
$[\eta\wedge]$	$\Gamma \vdash <\mathbf{p}_1(\mathrm{c}{:}\mathrm{A}\wedge\mathrm{B}),\mathbf{p}_2(\mathrm{c}{:}\mathrm{A}\wedge\mathrm{B})> = \mathrm{c}\ [: \mathrm{A} \wedge \mathrm{B}]$,	if $\Gamma \vdash \mathrm{c} : \mathrm{A} \wedge \mathrm{B}$,
$[\beta\forall]$	$\Gamma \vdash (!\mathrm{u}.\mathrm{a}\llbracket \mathrm{u}\rrbracket)[\mathbf{t}] = \mathrm{a}\llbracket \mathrm{u}{:=}\mathbf{t}\rrbracket\ [: \mathrm{A}\llbracket \mathrm{u}{:=}\mathbf{t}\rrbracket]$,	if $\Gamma \Vdash \mathbf{t} :: \mathbf{U}$, and $\Gamma[\mathrm{u}{:}\mathbf{U}] \vdash \mathrm{a}\llbracket \mathrm{u}\rrbracket : \mathrm{A}\llbracket \mathrm{u}\rrbracket$,
$[\eta\forall]$	$\Gamma \vdash !\mathrm{u}.\mathrm{f}[\mathrm{u}] = \mathrm{f}\ [: \forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket]$,	if $\Gamma \vdash \mathrm{f} : \forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket$, $[\mathrm{u} \notin \mathrm{FV}_u(\mathrm{f})]$.

2.2 "Prime" reductio rules. If A is a "prime" formula,

$[\oint \gamma]_0$ $\Gamma \vdash \gamma \mathrm{x}{:}\neg\mathrm{A}.\mathrm{f}\llbracket \mathrm{x}\rrbracket(\mathrm{x}(\gamma \mathrm{y}{:}\neg\mathrm{A}.\mathrm{e}\llbracket \mathrm{x},\mathrm{y}\rrbracket)) = \gamma \mathrm{z}{:}\neg\mathrm{A}.\mathrm{f}\llbracket \mathrm{z}\rrbracket(\mathrm{e}\llbracket \mathrm{z},\mathrm{z}\rrbracket)\ [: \mathrm{A}]$,
if $\Gamma[\mathrm{x}{:}\neg\mathrm{A}] \vdash \mathrm{f}\llbracket \mathrm{x}\rrbracket : \top$, $\Gamma[\mathrm{x}{:}\neg\mathrm{A}][\mathrm{y}{:}\neg\mathrm{A}] \vdash \mathrm{e}\llbracket \mathrm{x},\mathrm{y}\rrbracket : \bot$, **where** $\mathrm{e}\llbracket \mathrm{z},\mathrm{z}\rrbracket \equiv \mathrm{e}\llbracket \mathrm{x}{:=}\mathrm{z}\rrbracket\llbracket \mathrm{y}{:=}\mathrm{z}\rrbracket$,

$[\eta \to \gamma]_0$ $\Gamma \vdash \gamma \mathrm{x}{:}\neg\mathrm{A}.\mathrm{x}(\mathrm{f}) = \mathrm{f}\ [: \mathrm{A}]$, if $\Gamma \vdash \mathrm{f} : \mathrm{A}$, $[\mathrm{x} \notin \mathrm{FV}_\lambda(\mathrm{f})]$,

2.3 Congruence rules.

$[\rho]$	$\Gamma \vdash \mathrm{a} = \mathrm{a}\ [: \mathrm{A}]$,	if $\Gamma \vdash \mathrm{a} : \mathrm{A}$,
$[\sigma]$	$\Gamma \vdash \mathrm{a} = \mathrm{b}\ [: \mathrm{A}]$,	if $\Gamma \vdash \mathrm{b} = \mathrm{a} : \mathrm{A}$,
$[\tau]$	$\Gamma \vdash \mathrm{a} = \mathrm{c}\ [: \mathrm{A}]$,	if $\Gamma \vdash \mathrm{a} = \mathrm{b} : \mathrm{A}$, and $\Gamma \vdash \mathrm{b} = \mathrm{c} : \mathrm{A}$,
$[\xi \to \lambda]$	$\Gamma \vdash \lambda \mathrm{x}{:}\mathrm{A}.\mathrm{b}_1\llbracket \mathrm{x}\rrbracket = \lambda \mathrm{x}{:}\mathrm{A}.\mathrm{b}_2\llbracket \mathrm{x}\rrbracket\ [: \mathrm{A} \to \mathrm{B}]$,	if $\Gamma[\mathrm{x}{:}\mathrm{A}] \vdash \mathrm{b}_1\llbracket \mathrm{x}\rrbracket = \mathrm{b}_2\llbracket \mathrm{x}\rrbracket : \mathrm{B}$,
$[\xi \to \gamma]_0$	$\Gamma \vdash \gamma \mathrm{x}{:}\neg\mathrm{A}.\mathrm{e}_1\llbracket \mathrm{x}\rrbracket = \gamma \mathrm{x}{:}\neg\mathrm{A}.\mathrm{e}_2\llbracket \mathrm{x}\rrbracket\ [: \mathrm{A}]$,	if $\Gamma[\mathrm{x}{:}\neg\mathrm{A}] \vdash \mathrm{e}_1\llbracket \mathrm{x}\rrbracket = \mathrm{e}_2\llbracket \mathrm{x}\rrbracket : \bot$, for A "prime",
$[\mu \to]$	$\Gamma \vdash \mathrm{f}(\mathrm{a}) = \mathrm{g}(\mathrm{a})\ [: \mathrm{B}]$,	if $\Gamma \vdash \mathrm{f} = \mathrm{g} : \mathrm{A} \to \mathrm{B}$, and $\Gamma \vdash \mathrm{a} : \mathrm{A}$,
$[\nu \to]$	$\Gamma \vdash \mathrm{f}(\mathrm{a}) = \mathrm{f}(\mathrm{b})\ [: \mathrm{B}]$,	if $\Gamma \vdash \mathrm{a} = \mathrm{b} : \mathrm{A}$, and $\Gamma \vdash \mathrm{f} : \mathrm{A} \to \mathrm{B}$,
$[\xi\wedge]$	$\Gamma \vdash <\mathrm{a}_1{:}\mathrm{A},\mathrm{b}_1{:}\mathrm{B}> = <\mathrm{a}_2{:}\mathrm{A},\mathrm{b}_2{:}\mathrm{B}>\ [: \mathrm{A} \wedge \mathrm{B}]$,	if $\Gamma \vdash \mathrm{a}_1 = \mathrm{a}_2{:}\ \mathrm{A}$, and $\Gamma \vdash \mathrm{b}_1 = \mathrm{b}_2{:}\ \mathrm{B}$,
$[\nu\wedge_1]$	$\Gamma \vdash \mathbf{p}_1(\mathrm{f}{:}\mathrm{A}\wedge\mathrm{B}) = \mathbf{p}_1(\mathrm{g}{:}\mathrm{A}\wedge\mathrm{B})\ [: \mathrm{A}]$,	if $\Gamma \vdash \mathrm{f} = \mathrm{g} : \mathrm{A} \wedge \mathrm{B}$,
$[\nu\wedge_2]$	$\Gamma \vdash \mathbf{p}_2(\mathrm{f}{:}\mathrm{A}\wedge\mathrm{B}) = \mathbf{p}_2(\mathrm{g}{:}\mathrm{A}\wedge\mathrm{B})\ [: \mathrm{B}]$,	if $\Gamma \vdash \mathrm{f} = \mathrm{g} : \mathrm{A} \wedge \mathrm{B}$,
$[\xi\forall]$	$\Gamma \vdash !\mathrm{u}.\mathrm{a}_1\llbracket \mathrm{u}\rrbracket = !\mathrm{u}.\mathrm{a}_2\llbracket \mathrm{u}\rrbracket\ [: \forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket]$,	if $\Gamma[\mathrm{u}{:}\mathbf{U}] \vdash \mathrm{a}_1\llbracket \mathrm{u}\rrbracket = \mathrm{a}_2\llbracket \mathrm{u}\rrbracket : \mathrm{A}\llbracket \mathrm{u}\rrbracket$,
$[\mu\forall]$	$\Gamma \vdash \mathrm{f}[\mathbf{t}] = \mathrm{g}[\mathbf{t}]\ [: \mathrm{A}\llbracket \mathrm{u}{:=}\mathbf{t}\rrbracket]$,	if $\Gamma \vdash \mathrm{f} = \mathrm{g} : \forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket$, **where** $\Gamma \Vdash \mathbf{t} :: \mathbf{U}$.

Note. According to the previous proof-context conventions, the assumption on contexts $\Gamma[\mathrm{x}{:}\mathrm{C}]$ (resp. $\Gamma[\mathrm{u}{:}\mathbf{U}]$) is that $[\mathrm{x}{:}\mathrm{C}]$ (resp. $[\mathrm{u}{:}\mathbf{U}]$) does not occur in Γ. Also, a "**U**-context" Γ_u contains only **U**-parameters, i.e., it must be of the form $[\mathrm{u}_1{:}\mathbf{U}]\ldots[\mathrm{u}_m{:}\mathbf{U}]$ (so, "$\Gamma_u \Vdash \mathbf{t} :: \mathbf{U}$" means that the **U**-term **t** contains possibly free **U**-variables that are among the **U**-parameters of Γ_u).

Of course, the congruence rules of $\lambda\gamma_0$**CQ** could have been stated, more economically, in terms of syntactic proof-environments.

Remark (*Folklore*: $\lambda\pi!$). It is easy to see that the γ-free part of $\lambda\gamma_0$**CQ** is a proper – in fact, conservative – extension, $\lambda\pi!$ say, of the ordinary typed λ-calculus λ^τ. At a closer look, $\lambda\pi!$ is, essentially, a "pure" *Automath* system (without facilities for systematic abbreviations and module-writing, i.e., **Aut**-"books"), viz. the "pure" part of Zucker's [77] **Aut**-Π. It is also a proper part of the first-order fragment of Martin-Löf's [84] type theory. Proof-theoretically, $\lambda\pi!$ is the same thing as the [⊥,→,∧,∀]-fragment of the [proof-theory of] *Minimalkalkül*. In particular, $\lambda\pi!$ is known to be *Post-consistent*. Indeed, where $\mathbf{Cons}(\lambda)$ means "λ is Post consistent", one has easily $\mathbf{Cons}(\lambda^\tau) \iff \mathbf{Cons}(\lambda\pi) \iff \mathbf{Cons}(\lambda\pi!)$, where $\lambda\pi$ – the "typed λ-calculus with surjective pairing" – is the quantifier-free fragment of $\lambda\pi!$. (A similar statement holds for **SN** [= strong normalization] properties. This is well-known intuitionistic and/or *Automath* folklore; cf., e.g., [van Daalen 80], [Troelstra 86].)

As an immediate consequence from definitions, we have a

Lemma (*General reductio: stratification*). The following stratification rule is derivable in $\lambda\gamma_0\mathbf{CQ}$ (actually, in $\vdash_0[\mathbf{CQ}]$), for all formulas A in $[\bot,\to,\wedge,\forall]$:

$(\to i\gamma)$ $\Gamma[\mathrm{x{:}\neg A}] \vdash \mathrm{e}\llbracket \mathrm{x} \rrbracket : \bot \Rightarrow \Gamma \vdash \gamma\mathrm{x{:}\neg A.e}\llbracket \mathrm{x} \rrbracket : \mathrm{A}$.

Proof. By induction on the structure of A, using $(\to i\gamma)_0$ and the definition of $\gamma\mathrm{x{:}\neg A.e}\llbracket \mathrm{x} \rrbracket$, for $\mathrm{A} := \bot$, $(\mathrm{B}\to\mathrm{C})$, $(\mathrm{B}\wedge\mathrm{C})$, $(\forall \mathrm{u.B}\llbracket \mathrm{u} \rrbracket)$. □

Notation ($\vdash_{\&}[\mathbf{CQ}]$ *and* $\vdash[\mathbf{CQ}]$).

In what follows, the Boolean proof-term stratification based on $[\bot,\to,\wedge,\forall]$, with *primitive* γ-forms $\gamma\mathrm{x{:}\neg A.e}\llbracket \mathrm{x} \rrbracket$, for all formulas A in $[\bot,\to,\wedge,\forall]$, and the rules of $\vdash_0[\mathbf{CQ}]$, with $(\to i\gamma)_0$ replaced by $(\to i\gamma)$, is referred to by $\vdash_{\&}[\mathbf{CQ}]$. The fragment of $\vdash_{\&}[\mathbf{CQ}]$, without $\wedge$ and $\wedge$-proof-primitives (defined for formulas in $[\bot,\to,\forall]$ alone), is referred to next by $\vdash[\mathbf{CQ}]$.

"Type-checking" proof-terms: the sub-proof property and proof-categoricity for $\vdash_0[\mathbf{CQ}]$ *and* $\vdash_{\&}[\mathbf{CQ}]$. The "type-checking" of a Boolean p-term, i.e., the verification of its "correctness" relative to $\vdash_0[\mathbf{CQ}]$, the stratification of $\lambda\gamma_0\mathbf{CQ}$, corresponds, in traditional terms, to "theorem proving". The theoretical basis of proof-term "type-checking" is in the fact that **CQ** is *proof-categorical relative to the type-theoretic presentation* $\vdash_0[\mathbf{CQ}]$.

Recall that "$\Gamma \vdash \mathrm{c}$" is shorthand for "$\Gamma \vdash \mathrm{c} : \mathrm{C}$, for some C". Where $\vdash$ stands for derivability in $\vdash_0[\mathbf{CQ}]$, we have the following

Lemma (*Inversion Lemma for* $\vdash_0[\mathbf{CQ}]$).

(111) $\Gamma \vdash \lambda\mathrm{x{:}A.b}\llbracket \mathrm{x} \rrbracket \Rightarrow \Gamma \vdash \lambda\mathrm{x{:}A.b}\llbracket \mathrm{x} \rrbracket : (\mathrm{A} \to \mathrm{B})$, for some B,
$(112)_0$ $\Gamma \vdash \gamma\mathrm{x{:}C.e}\llbracket \mathrm{x} \rrbracket \Rightarrow \mathrm{C} \equiv \neg\mathrm{A}$, and $\Gamma \vdash \gamma\mathrm{x{:}\neg A.e}\llbracket \mathrm{x} \rrbracket : \mathrm{A}$, for some "prime" formula A,
(12) $\Gamma \vdash <\mathrm{a{:}A,b{:}B}> \Rightarrow \Gamma \vdash <\mathrm{a{:}A,b{:}B}> : \mathrm{A} \wedge \mathrm{B}$,
(13) $\Gamma \vdash !\mathrm{u.a}\llbracket \mathrm{u} \rrbracket \Rightarrow \Gamma \vdash !\mathrm{u.a}\llbracket \mathrm{u} \rrbracket : \forall\mathrm{u.A}\llbracket \mathrm{u} \rrbracket$, for some $\mathrm{A}\llbracket \mathrm{u} \rrbracket$, with, possibly, $\mathrm{u} \in \mathrm{FV}_u(\mathrm{A}\llbracket \mathrm{u} \rrbracket)$,
(21) $\Gamma \vdash \mathrm{f(a)} \Rightarrow \Gamma \vdash \mathrm{f} : \mathrm{A} \to \mathrm{B}, \Gamma \vdash \mathrm{a} : \mathrm{A}$, for some A, B, such that $\Gamma \vdash \mathrm{f(a)} : \mathrm{B}$,
(221) $\Gamma \vdash \mathbf{p}_1(\mathrm{c{:}A\wedge B}) \Rightarrow \Gamma \vdash \mathbf{p}_1(\mathrm{c{:}A\wedge B}) : \mathrm{A}$,
(222) $\Gamma \vdash \mathbf{p}_2(\mathrm{c{:}A\wedge B}) \Rightarrow \Gamma \vdash \mathbf{p}_2(\mathrm{c{:}A\wedge B}) : \mathrm{B}$,
(23) $\Gamma \vdash \mathrm{f}[\mathbf{t}] \Rightarrow \Gamma \vdash \mathrm{f}[\mathbf{t}] : \mathrm{A}\llbracket \mathrm{u}{:=}\mathbf{t} \rrbracket$, for some $\mathrm{A}\llbracket \mathrm{u} \rrbracket$, and $\Gamma \Vdash \mathbf{t} :: \mathbf{U}$,
where $\mathrm{u} \in \mathrm{FV}_u(\mathrm{A}\llbracket \mathrm{u} \rrbracket)$, with $\mathbf{t}$ free for u in $\mathrm{A}\llbracket \mathrm{u} \rrbracket$.

Proof. By the stratification rules of $\vdash_0[\mathbf{CQ}]$. □

For $\vdash_{\&}[\mathbf{CQ}]$, we have, in general, a

Corollary (*Inversion Lemma for* $\vdash_{\&}[\mathbf{CQ}]$). The *Inversion Lemma* holds for $\vdash_{\&}[\mathbf{CQ}]$, too, with, moreover,

(112) $\Gamma \vdash_{\&} \gamma\mathrm{x{:}C.e}\llbracket \mathrm{x} \rrbracket \Rightarrow \mathrm{C} \equiv \neg\mathrm{A}$, for some A, and $\Gamma \vdash_{\&} \gamma\mathrm{x{:}\neg A.e}\llbracket \mathrm{x} \rrbracket : \mathrm{A}$,

(A in $[\bot,\to,\wedge,\forall]$), in place of $(112)_0$.

Proof. As above, using also $(\to i\gamma)$, i.e., the stratification rules of $\vdash_{\&}[\mathbf{CQ}]$. □

This yields a *rigorous substitute* for the traditional talk about a "subformula-property". Indeed, where $\vdash$ stands for "correctness" [derivability] in $\vdash_0[\mathbf{CQ}]$, we have a

Theorem (*Sub-proof Correctness for* $\vdash_0[\mathbf{CQ}]$).

(111) $\Gamma \vdash \lambda\mathrm{x{:}A.a}\llbracket \mathrm{x} \rrbracket : (\mathrm{A}\to\mathrm{B}) \Rightarrow \Gamma[\mathrm{x{:}A}] \vdash \mathrm{a}\llbracket \mathrm{x} \rrbracket : \mathrm{B}$,
$(112)_0$ $\Gamma \vdash \gamma\mathrm{x{:}\neg A.e}\llbracket \mathrm{x} \rrbracket : \mathrm{A} \Rightarrow \Gamma[\mathrm{x{:}\neg A}] \vdash \mathrm{e}\llbracket \mathrm{x} \rrbracket : \bot$, for any "prime" A,
(12) $\Gamma \vdash <\mathrm{a{:}A,b{:}B}> : (\mathrm{A}\wedge\mathrm{B}) \Rightarrow \Gamma \vdash \mathrm{a} : \mathrm{A}$, and $\Gamma \vdash \mathrm{b} : \mathrm{B}$,
(13) $\Gamma \vdash !\mathrm{u.a}\llbracket \mathrm{u} \rrbracket : \forall\mathrm{u.A}\llbracket \mathrm{u} \rrbracket \Rightarrow \Gamma[\mathrm{u}{:}\mathbf{U}] \vdash \mathrm{a}\llbracket \mathrm{u} \rrbracket : \mathrm{A}\llbracket \mathrm{u} \rrbracket$, u fresh for Γ,
(21) $\Gamma \vdash \mathrm{f(c)} : \mathrm{A} \Rightarrow \Gamma \vdash \mathrm{f} : \mathrm{C} \to \mathrm{A}$, and $\Gamma \vdash \mathrm{c} : \mathrm{C}$, for some C,
(221) $\Gamma \vdash \mathbf{p}_1(\mathrm{c{:}A\wedge B}) : \mathrm{A} \Rightarrow \Gamma \vdash \mathrm{c} : \mathrm{A} \wedge \mathrm{B}$,
(222) $\Gamma \vdash \mathbf{p}_2(\mathrm{c{:}A\wedge B}) : \mathrm{B} \Rightarrow \Gamma \vdash \mathrm{c} : \mathrm{A} \wedge \mathrm{B}$,

(23) $\Gamma \vdash f[\mathbf{t}] : A[\![u{:=}\mathbf{t}]\!]$ $\Rightarrow \Gamma \vdash f : \forall v.A[\![u{:=}v]\!]$, and $\Gamma \Vdash \mathbf{t} :: \mathbf{U}$,
$\mathbf{t}$ free for u in $A[\![u]\!]$,
v fresh for Γ, not free in $A[\![u{:=}\mathbf{t}]\!]$, and f.

Proof. By the *Inversion Lemma* for $\vdash_0[\mathbf{CQ}]$. □

Corollary (*Sub-proof Correctness for* $\vdash_\&[\mathbf{CQ}]$). The *Sub-proof Correctness Theorem* holds for $\vdash_\&[\mathbf{CQ}]$, with, also, for all formulas A in $[\bot,\to,\wedge,\forall]$,

(112) $\Gamma \vdash_\& \gamma x{:}\neg A.e[\![x]\!] : A \;\Rightarrow \Gamma[x{:}\neg A] \vdash_\& e[\![x]\!] : \bot$,

in place of $(112)_0$.

Proof. By the *Inversion Lemmas* above. □

Remark (*Type-checking algorithms for* **CQ**-*proofs*). The *Sub-proof Correctness Theorems* guarantee the existence of a *type-checking algorithm* for first-order classical logic proof(-term)s. This does not depend on considerations about proof-reduction and/or proof-equality in **CQ**.

Corollary (*Global Sub-proof Correctness for* $\vdash_0[\mathbf{CQ}]$). Let a, b be proof-terms of $\vdash_0[\mathbf{CQ}]$ such that b is a subterm of a. If $\Gamma \vdash$ a then $\Gamma' \vdash$ b, for some proof-context Γ'.

Proof. By induction on the subterm structure of a, using the above. □

Theorem (*Proof-categoricity relative to* $\vdash_0[\mathbf{CQ}]$: [**UT-CQ**]). If $\Gamma \vdash a : A_1$ and $\Gamma \vdash a : A_2$ then $A_1 \equiv A_2$.

Proof. By induction on the structure of a, using the *Sub-proof Correctness Theorem* for $\vdash_0[\mathbf{CQ}]$. □

In is clear that the last two statements hold for $\vdash_\&[\mathbf{CQ}]$ – and so $\vdash[\mathbf{CQ}]$ –, too. In other words, classical logic **CQ** is *proof-categorical relative to the type-theoretic presentations/stratifications* $\vdash_0[\mathbf{CQ}]$, $\vdash_\&[\mathbf{CQ}]$, resp.

The λγ-calculus $\lambda\gamma_\&\mathbf{CQ}$. In the above, the rule $[\eta \to \gamma]_0$ is an *extensionality rule* for "prime" γ-abstractions, whereas $[\oint \gamma]_0$ is a *diagonalization rule* for "prime" uses of γ [*reductio ad absurdum*] (cf., e.g., the "commuting" behavior of the Heyting [∨,∃]-selectors). In $\lambda\gamma_0\mathbf{CQ}$, these rules can be seen to hold for *every* use of γ, as defined in the proof-term syntax, viz.

Lemma (*General reductio: basic equational behavior*). The following equations ([*reductio diagonalization*] and [*reductio extensionality*], resp.) are derivable in $\lambda\gamma_0\mathbf{CQ}$, for all formulas A in $[\bot,\to,\wedge,\forall]$:

$[\oint \gamma]$ $\Gamma \vdash \gamma x{:}\neg A.f[\![x]\!](x(\gamma y{:}\neg A.e[\![x,y]\!])) = \gamma z{:}\neg A.f[\![z]\!](e[\![z,z]\!])$ [: A],
if $\Gamma[x{:}\neg A] \vdash f[\![x]\!] : \top$, $\Gamma[x{:}\neg A][y{:}\neg A] \vdash e[\![x,y]\!] : \bot$, **where** $e[\![z,z]\!] \equiv e[\![x{:=}z]\!][\![y{:=}z]\!]$,
$[\eta \to \gamma]$ $\Gamma \vdash \gamma x{:}\neg A.x(f) = f$ [: A], if $\Gamma \vdash f : A$, $[x \notin FV_\lambda(f)]$.

Proof. In each case, by induction on the structure of A, using the "prime" rules $[\oint \gamma]_0$, $[\eta \to \gamma]_0$, as a basis of the induction, and the definition of $\gamma x{:}\neg A.e[\![x]\!]$, for A := ⊥, (B→C), (B∧C), $(\forall u.B[\![u]\!])$. □

Remark (*General γ-congruence*). The following congruence rule is derivable in $\lambda\gamma_0\mathbf{CQ}$, for all formulas A in $[\bot,\to,\wedge,\forall]$ [*exercise*]:

$[\xi \to \gamma]$ $\Gamma \vdash \gamma x{:}\neg A.e_1[\![x]\!] = \gamma x{:}\neg A.e_2[\![x]\!]$ [: A], if $\Gamma[x{:}\neg A] \vdash e_1[\![x]\!] = e_2[\![x]\!] : \bot$.

Remark (*The diagonalization rule*). The *diagonalization rule* $[\oint \gamma]$ can be also stated in terms of syntactic proof-environments, viz.

$\ll\!\oint \gamma\!\gg$ $\Gamma \vdash \gamma x{:}\neg A.\varphi\!\ll x(\gamma y{:}\neg A.c[\![x,y]\!]) \gg\; = \gamma z{:}\neg A.\varphi\!\ll c[\![x{:=}z]\!][\![y{:=}z]\!] \gg$ [: A],

for any syntactic proof-environment $\varphi\!\ll \bullet \gg$, with $\Gamma[x{:}\neg A] \vdash \varphi\!\ll x(\gamma y{:}\neg A.c[\![x,y]\!]) \gg\, : \bot$. As a *special case*, one has a *weak diagonalization* rule:

$[\oint_0\gamma]$ $\Gamma \vdash \gamma x{:}\neg A.x(\gamma y{:}\neg A.c[\![x,y]\!]) = \gamma z{:}\neg A.c[\![x{:=}z]\!][\![y{:=}z]\!]$ [: A], if $\Gamma[x{:}\neg A][y{:}\neg A] \vdash c[\![x,y]\!] : \bot$,

(which is, indeed, *weaker* than $[\oint \gamma]$, for one cannot identify consistently, in $\lambda\gamma_0\mathbf{CQ}$, any f such that $\Gamma \vdash f : \top \equiv (\bot \to \bot)$ with $\Omega \equiv \lambda x{:}\bot.x$). In the general case $\ll\!\oint \gamma\!\gg$, the stratification *proviso*

$$\Gamma[\mathrm{x}{:}\neg \mathrm{A}] \vdash \varphi\!\ll \mathrm{x}(\gamma \mathrm{y}{:}\neg\mathrm{A}.\mathrm{c}\llbracket \mathrm{x},\mathrm{y}\rrbracket) \gg,\ \varphi\!\ll \mathrm{c}\llbracket \mathrm{x}{:=}\mathrm{z}\rrbracket\llbracket \mathrm{y}{:=}\mathrm{z}\rrbracket \gg\ :\ \bot,$$

for any syntactic proof-environment $\varphi\!\ll \bullet \gg$, amounts to the fact that $\Gamma(\varphi) \vdash \mathrm{c}\llbracket \mathrm{x},\mathrm{y}\rrbracket : \bot$, for some proof-context $\Gamma(\varphi)$ extending $\Gamma[\mathrm{x}{:}\neg\mathrm{A}][\mathrm{y}{:}\neg\mathrm{A}]$ and depending possibly on $\varphi\!\ll \bullet \gg$.

Lemma (*Boolean reductio lifting*). The following equations are derivable in $\lambda\gamma_0$**CQ**, for all formulas A, B in $[\bot,\rightarrow,\wedge,\forall]$, and all p-terms e:

$[{}^h\beta\gamma \rightarrow]$ $\Gamma \vdash \gamma \mathrm{x}{:}\neg(\mathrm{A}{\rightarrow}\mathrm{B}).\mathrm{e}\llbracket \mathrm{x}\rrbracket = \lambda \mathrm{x}_0{:}\mathrm{A}.\gamma \mathrm{x}_1{:}\neg\mathrm{B}.\mathrm{e'}\llbracket \mathrm{x}_0,\mathrm{x}_1\rrbracket\ [:\ \mathrm{A} \rightarrow \mathrm{B}]$,
if $\Gamma[\mathrm{x}{:}\neg(\mathrm{A}{\rightarrow}\mathrm{B})] \vdash \mathrm{e}\llbracket \mathrm{x}\rrbracket : \bot$, **where** $\mathrm{e'}\llbracket \mathrm{x}_0,\mathrm{x}_1\rrbracket \equiv \mathrm{e}\llbracket \mathrm{x}{:=}\lambda \mathrm{z}{:}(\mathrm{A}{\rightarrow}\mathrm{B}).\mathrm{x}_1(\mathrm{z}(\mathrm{x}_0))\rrbracket$,

$[{}^h\beta\gamma\wedge]$ $\Gamma \vdash \gamma \mathrm{x}{:}\neg(\mathrm{A}{\wedge}\mathrm{B}).\mathrm{e}\llbracket \mathrm{x}\rrbracket = \ <\mathrm{a}{:}\mathrm{A},\mathrm{b}{:}\mathrm{B}>\ [:\ \mathrm{A} \wedge \mathrm{B}]$, if $\Gamma[\mathrm{x}{:}\neg(\mathrm{A}{\wedge}\mathrm{B})] \vdash \mathrm{e}\llbracket \mathrm{x}\rrbracket : \bot$, **where**
$\mathrm{a} \equiv \gamma \mathrm{x}_1{:}\neg\mathrm{A}.\mathrm{e}\llbracket \mathrm{x}{:=}\lambda \mathrm{z}{:}(\mathrm{A}{\wedge}\mathrm{B}).\mathrm{x}_1(\mathbf{p}_1(\mathrm{z}{:}\mathrm{A}{\wedge}\mathrm{B}))\rrbracket$, and
$\mathrm{b} \equiv \gamma \mathrm{x}_2{:}\neg\mathrm{B}.\mathrm{e}\llbracket \mathrm{x}{:=}\lambda \mathrm{z}{:}(\mathrm{A}{\wedge}\mathrm{B}).\mathrm{x}_2(\mathbf{p}_2(\mathrm{z}{:}\mathrm{A}{\wedge}\mathrm{B}))\rrbracket$,

$[{}^h\beta\gamma\forall]$ $\Gamma \vdash \gamma \mathrm{x}{:}\neg(\forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket).\mathrm{e}\llbracket \mathrm{x}\rrbracket = \,!\mathrm{u}.\gamma \mathrm{x}_1{:}\neg\mathrm{A}\llbracket \mathrm{u}\rrbracket.\mathrm{e'}\llbracket \mathrm{u},\mathrm{x}_1\rrbracket\ [:\ \forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket]$,
if $\Gamma[\mathrm{x}{:}\neg(\forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket)] \vdash \mathrm{e}\llbracket \mathrm{x}\rrbracket : \bot$, **where** $\mathrm{e'}\llbracket \mathrm{u},\mathrm{x}_1\rrbracket \equiv \mathrm{e}\llbracket \mathrm{x}{:=}\lambda \mathrm{z}{:}(\forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket).\mathrm{x}_1(\mathrm{z}[\mathrm{u}])\rrbracket$.

Proof. From the definition of $\gamma \mathrm{x}{:}\neg\mathrm{A}.\mathrm{e}\llbracket \mathrm{x}\rrbracket$, for $\mathrm{A} := \bot$, $(\mathrm{B}{\rightarrow}\mathrm{C})$, $(\mathrm{B}{\wedge}\mathrm{C})$, $(\forall \mathrm{u}.\mathrm{B}\llbracket \mathrm{u}\rrbracket)$, using $(\rightarrow\! i\gamma)$ in order to insure the stratification conditions. □

Theorem (*The Boolean normal reductio rules*). The following equations are derivable in $\lambda\gamma_0$**CQ**, for all formulas A, B in $[\bot,\rightarrow,\wedge,\forall]$, all p-terms e, a, and all **U**-terms **t**:

$[\beta\gamma\bot]$ $\Gamma \vdash \gamma \mathrm{x}{:}\top.\mathrm{e}\llbracket \mathrm{x}\rrbracket = \mathrm{e}\llbracket \mathrm{x}{:=}\Omega\rrbracket\ [:\ \bot]$, if $\Gamma[\mathrm{x}{:}\top] \vdash \mathrm{e}\llbracket \mathrm{x}\rrbracket : \bot$,

$[\beta\gamma \rightarrow]$ $\Gamma \vdash (\gamma \mathrm{x}{:}\neg(\mathrm{A}{\rightarrow}\mathrm{B}).\mathrm{e}\llbracket \mathrm{x}\rrbracket)\mathrm{a} = \gamma \mathrm{x}{:}\neg\mathrm{B}.\mathrm{e}\llbracket \mathrm{x}{:=}\lambda \mathrm{z}{:}(\mathrm{A}{\rightarrow}\mathrm{B}).\mathrm{x}(\mathrm{z}(\mathrm{a}))\rrbracket\ [:\ \mathrm{B}]$,
if $\Gamma \vdash \mathrm{a} : \mathrm{A}$, and $\Gamma[\mathrm{x}{:}\neg(\mathrm{A}{\rightarrow}\mathrm{B})] \vdash \mathrm{e}\llbracket \mathrm{x}\rrbracket : \bot$,

$[\beta\gamma\wedge_1]$ $\Gamma \vdash \mathbf{p}_1((\gamma \mathrm{x}{:}\neg(\mathrm{A}{\wedge}\mathrm{B}).\mathrm{e}\llbracket \mathrm{x}\rrbracket){:}(\mathrm{A}{\wedge}\mathrm{B})) = \gamma \mathrm{x}_1{:}\neg\mathrm{A}.\mathrm{e}_1\llbracket \mathrm{x}_1\rrbracket\ [:\ \mathrm{A}]$,
if $\Gamma[\mathrm{x}{:}\neg(\mathrm{A}{\wedge}\mathrm{B})] \vdash \mathrm{e}\llbracket \mathrm{x}\rrbracket : \bot$, **where** $\mathrm{e}_1\llbracket \mathrm{x}_1\rrbracket \equiv \mathrm{e}\llbracket \mathrm{x}{:=}\lambda \mathrm{z}{:}(\mathrm{A}{\wedge}\mathrm{B}).\mathrm{x}_1(\mathbf{p}_1(\mathrm{z}))\rrbracket$,

$[\beta\gamma\wedge_2]$ $\Gamma \vdash \mathbf{p}_2((\gamma \mathrm{x}{:}\neg(\mathrm{A}{\wedge}\mathrm{B}).\mathrm{e}\llbracket \mathrm{x}\rrbracket){:}(\mathrm{A}{\wedge}\mathrm{B})) = \gamma \mathrm{x}_2{:}\neg\mathrm{B}.\mathrm{e}_2\llbracket \mathrm{x}_2\rrbracket\ [:\ \mathrm{B}]$,
if $\Gamma[\mathrm{x}{:}\neg(\mathrm{A}{\wedge}\mathrm{B})] \vdash \mathrm{e}\llbracket \mathrm{x}\rrbracket : \bot$, **where** $\mathrm{e}_2\llbracket \mathrm{x}_2\rrbracket \equiv \mathrm{e}\llbracket \mathrm{x}{:=}\lambda \mathrm{z}{:}(\mathrm{A}{\wedge}\mathrm{B}).\mathrm{x}_2(\mathbf{p}_2(\mathrm{z}))\rrbracket$,

$[\beta\gamma\forall]$ $\Gamma \vdash (\gamma \mathrm{x}{:}\neg(\forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket).\mathrm{e}\llbracket \mathrm{x}\rrbracket)[\mathbf{t}] = \gamma \mathrm{x}{:}\neg\mathrm{A}\llbracket \mathrm{u}{:=}\mathbf{t}\rrbracket.\mathrm{e'}\llbracket \mathrm{x}\rrbracket\ [:\ \mathrm{A}\llbracket \mathrm{u}{:=}\mathbf{t}\rrbracket]$,
if $\Gamma \Vdash \mathbf{t} :: \mathbf{U}$, and $\Gamma[\mathrm{x}{:}\neg(\forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket)] \vdash \mathrm{e}\llbracket \mathrm{x}\rrbracket : \bot$, **where** $\mathrm{e'}\llbracket \mathrm{x}\rrbracket \equiv \mathrm{e}\llbracket \mathrm{x}{:=}\lambda \mathrm{z}{:}(\forall \mathrm{u}.\mathrm{A}\llbracket \mathrm{u}\rrbracket).\mathrm{x}(\mathrm{z}[\mathbf{t}])\rrbracket$.

Proof. From the ^{h}-rules above, using the β-rules of $\lambda\gamma_0$**CQ**. □

Remark (*The* $\lambda\gamma_{(\&)}$**CQ**- *and* ${}^h\lambda\gamma_{(\&)}$**CQ**-*calculi/theories*). So, a (stratification/equationally) equivalent formulation of $\lambda\gamma_0$**CQ** could be the theory $\lambda\gamma_\&$**CQ**, using the full γ-syntax, based on $\vdash_\&$[**CQ**] [with a *primitive stratification* rule $(\rightarrow\! i\gamma)$ in place of the "prime" rule $(\rightarrow\! i\gamma)_0$ and] with, as γ-postulates:

- the *normal* $\beta\gamma$-*rules*: $[\beta\gamma\bot]$, $[\beta\gamma \rightarrow]$, $[\beta\gamma\wedge_1]$, $[\beta\gamma\wedge_2]$, $[\beta\gamma\forall]$,
- the *diagonalization rule* $[\oint \gamma]$,
- the *extensionality rule* $[\eta \rightarrow \gamma]$, and
- the *general* γ-*congruence* $[\xi \rightarrow \gamma]$, above.

In view of the η-rules of $\lambda\gamma_\&$**CQ**, one can reformulate equivalently the latter one, in the same syntax, as a theory ${}^h\lambda\gamma_\&$**CQ**, with ^{h}-*rules* ($[{}^h\beta\gamma\bot]$, $[{}^h\beta\gamma \rightarrow]$, $[{}^h\beta\gamma\wedge]$, $[{}^h\beta\gamma\forall]$) as postulates, in place of the *normal* $\beta\gamma$-*rules* of $\lambda\gamma_\&$**CQ**. The sub-theories of $\lambda\gamma_\&$**CQ**, ${}^h\lambda\gamma_\&$**CQ**, resp. defined on $\vdash$[**CQ**], without $\wedge$-types and $\wedge$-proof-primitives, are referred to next by $\lambda\gamma$**CQ** and ${}^h\lambda\gamma$**CQ**, resp.

In particular, $[{}^h\beta\gamma \rightarrow]$ yields the kernel of the Glivenko [28,29] "negative" translation, as extended to the *proofs themselves*:

Corollary (*The Glivenko "negative" proof-translation*).

$[{}^h\beta\gamma\neg]$ $\Gamma \vdash \gamma \mathrm{x}{:}(\neg\neg\mathrm{A}).\mathrm{e}\llbracket \mathrm{x}\rrbracket = \lambda \mathrm{x}_0{:}\mathrm{A}.\mathrm{e}\llbracket \mathrm{x}{:=}\lambda \mathrm{z}{:}(\neg\mathrm{A}).\mathrm{z}(\mathrm{x}_0)\rrbracket\ [:\ \neg\mathrm{A}]$, if $\Gamma[\mathrm{x}{:}(\neg\neg\mathrm{A})] \vdash \mathrm{e}\llbracket \mathrm{x}\rrbracket : \bot$.

Proof. From $[{}^h\beta\gamma \rightarrow]$ and $[\beta\gamma\bot]$. □

This suggests a straightforward way of showing *Post-consistency* for the classical proof-calculus $\lambda\gamma_0$**CQ**, [i.e., **Cons**($\lambda\gamma_0$**CQ**), whence **Cons**($\lambda\gamma_\&$**CQ**), by equational equivalence, and so **Cons**($\lambda\gamma$**CQ**)].

With $\vdash$ used ambiguously for derivability in $\lambda\gamma_0\mathbf{CQ}$ and/or $\lambda\pi!$, we have the following

Definition (*The Glivenko "negative" translation*).

Define a map $(\dots)^N$ from proof-statements $\Gamma \vdash \mathrm{a} : \mathrm{A}$ of $\lambda\gamma_0\mathbf{CQ}$ to proof-statements $(\Gamma)^N \vdash (\mathrm{a})^N : (\mathrm{A})^N$ of $\lambda\pi!$ by:

- $(\mathbf{t})^N \equiv \mathbf{t}$, for any **U**-term $\mathbf{t}$,
- $(\mathrm{A})^N \equiv \mathrm{A}[\![\mathtt{P}_1{:=}\neg\mathtt{P}_1]\!]\dots[\![\mathtt{P}_n{:=}\neg\mathtt{P}_n]\!]$, for any **H**-*atom* $\mathtt{P}_i$ $(1 \le i \le n)$, occurring in A,
- $(\Gamma)^N \equiv \Gamma_u \smile [\mathrm{x}_1{:}(\mathrm{A}_1)^N]\dots[\mathrm{x}_n{:}(\mathrm{A}_n)^N]$, for any proof-context $\Gamma := \Gamma_u \smile [\mathrm{x}_1{:}\mathrm{A}_1]\dots[\mathrm{x}_n{:}\mathrm{A}_n]$
- $(\mathrm{a})^N$, by induction on the structure of a (in $\vdash_0[\mathbf{CQ}]$),
  - $(\mathrm{x})^N \equiv \mathrm{x}$,
  - $(\lambda\mathrm{x}{:}\mathrm{A}.\mathrm{a}[\![\mathrm{x}]\!])^N \equiv \lambda\mathrm{x}{:}(\mathrm{A})^N.(\mathrm{a}[\![\mathrm{x}]\!])^N$,
  - $(\mathrm{fa})^N \equiv (\mathrm{f})^N(\mathrm{a})^N$,
  - $(\gamma\mathrm{x}{:}\neg\mathrm{A}.\mathrm{e}[\![\mathrm{x}]\!])^N \equiv \lambda\mathrm{x}_0{:}\mathrm{A}.(\mathrm{e}[\![\mathrm{x}]\!])^N[\![\mathrm{x}{:=}\lambda\mathrm{z}{:}\neg\mathrm{A}.\mathrm{z}(\mathrm{x}_0)]\!]$, for "prime" formulas A,
  - $(<\mathrm{a}{:}\mathrm{A},\ \mathrm{b}{:}\mathrm{B}>)^N \equiv\ <(\mathrm{a})^N{:}(\mathrm{A})^N,\ (\mathrm{b})^N{:}(\mathrm{A})^N>$,
  - $(\mathbf{p}_1(\mathrm{f}{:}\mathrm{A}\wedge\mathrm{B}))^N \equiv \mathbf{p}_1((\mathrm{f})^N{:}(\mathrm{A}\wedge\mathrm{B})^N)$,
  - $(\mathbf{p}_2(\mathrm{f}{:}\mathrm{A}\wedge\mathrm{B}))^N \equiv \mathbf{p}_2((\mathrm{f})^N{:}(\mathrm{A}\wedge\mathrm{B})^N)$,
  - $(!\mathrm{u}.\mathrm{a}[\![\mathrm{u}]\!])^N \equiv\ !\mathrm{u}.(\mathrm{a}[\![\mathrm{u}]\!])^N$,
  - $(\mathrm{f}[\mathbf{t}])^N \equiv (\mathrm{f})^N([\mathbf{t}])^N$.

One checks easily the fact that $(\dots)^N$ is well-defined as a map [*exercise*]. From this, we get the expected "*Glivenko Lemmas*", by applying $(\dots)^N$ to (**CQ**-) proof-statements and proof-equations, resp.:

Lemma (V. I. Glivenko, 1928). $\Gamma \vdash \mathrm{a} : \mathrm{A}$ in $\lambda\gamma_0\mathbf{CQ} \Rightarrow (\Gamma)^N \vdash (\mathrm{a})^N : (\mathrm{A})^N$ in $\lambda\pi!$.

Proof. By induction on $\vdash_0[\mathbf{CQ}]$. $\square$

Lemma ($\lambda\pi! \supseteq (\lambda\gamma_0\mathbf{CQ})^N$). $\Gamma \vdash \mathrm{a}_1 = \mathrm{a}_2$ in $\lambda\gamma_0\mathbf{CQ} \Rightarrow (\Gamma)^N \vdash (\mathrm{a}_1)^N = (\mathrm{a}_2)^N$ in $\lambda\pi!$.

Proof. Immediate, for the $(\dots)^N$-images of the "prime" rules $[\oint\gamma]_0$, $[\eta\to\gamma]_0$, $[\xi\to\gamma]_0$ hold in $\lambda\pi!$. $\square$

In other words, $\lambda\pi!$ extends $\lambda\gamma_0\mathbf{CQ}$, *modulo* the Glivenko "negative" translation $(\dots)^N$. In fact, if $\Gamma \vdash \mathrm{a}_1 = \mathrm{a}_2 : \mathrm{A}$ holds in $\lambda\gamma_0\mathbf{CQ}$ then $(\Gamma)^N \vdash (\mathrm{a}_1)^N = (\mathrm{a}_2)^N : (\mathrm{A})^N$ holds in $\lambda\pi!$.

Theorem ($\mathbf{Cons}(\lambda\gamma_0\mathbf{CQ})$). $\lambda\gamma_0\mathbf{CQ}$ is Post-consistent.

Proof. The above gives $\mathbf{Cons}(\lambda\pi!) \Rightarrow \mathbf{Cons}(\lambda\gamma_0\mathbf{CQ})$ and we know that $\mathbf{Cons}(\lambda\pi!)$. [Actually, $\mathbf{Cons}(\lambda^\tau) \iff \mathbf{Cons}(\lambda\pi!)$ and $\mathbf{Cons}(\lambda^\tau)$.] $\square$

Corollary ($\mathbf{Cons}(\lambda\gamma_{(\&)}\mathbf{CQ})$). $\lambda\gamma_{(\&)}\mathbf{CQ}$ is Post-consistent.

Proof. $\lambda\gamma_0\mathbf{CQ}$ and $\lambda\gamma_\&\mathbf{CQ}$ are equationally equivalent and $\mathbf{Cons}(\lambda\gamma_0\mathbf{CQ})$, while $\lambda\gamma\mathbf{CQ}$ is a sub-theory of $\lambda\gamma_\&\mathbf{CQ}$. $\square$

In other words: *the first-order classical logic is proof-consistent*.

There is also a *type-free* variant of the consistency-proof for $\lambda\gamma_{(\&)}\mathbf{CQ}$ [Rezuş 90]. On the other hand, the "Glivenko" argument above can be easily transformed into a *standard model-construction* for $\lambda\gamma_\&\mathbf{CQ}$ [*exercise*]. (*Hint*. See, *mutatis mutandis*, [Rezuş 91].)

Conjecture (*Post-completeness for* $\lambda\gamma_{(0,\&)}\mathbf{CQ}$). $\lambda\gamma_{(0,\&)}\mathbf{CQ}$ is Post-complete.

That is to say, essentially: if $\mathbf{CQ} \Vdash \mathrm{A}$ [i.e., if A is provable in **CQ**] then the addition of any non-derivable *closed proof-equation* $\mathrm{a}_1 = \mathrm{a}_2$ (where $[\ ] \vdash \mathrm{a}_1, \mathrm{a}_2 : \mathrm{A}$) to $\lambda\gamma_{(0,\&)}\mathbf{CQ}$ – in the corresponding proof-language – would also make the resulting extension Post-inconsistent; viz., any two proofs of A should be equal, in the extended sense, whence "proof-irrelevance". The property referred to is, in fact, a "typed" analogue of the well-known *Böhm* [68] *saturation property* for the extensional *type-free* λ-calculus $\lambda\beta\eta\mathbf{K}$, where one cannot identify consistently any two normal combinators (in the "typed" case, the normali[zabili]ty condition is implied by stratifiability).

CHAPTER IV

THE PROOF-THEORY OF JOHANSSON'S MINIMALKALKÜL

Alternative Boolean proof-term stratifications. In order to be able to isolate proof-theoretically interesting fragments of **CQ**, one needs some further details on the alternative Boolean proof-term stratifications.

Remark (*The general positive selectors*). If the general positive selectors $\bigwedge_{\vdash}$, $\bigwedge_{\natural}$, $\bigwedge_{\cup}$ are defined as above, in terms of "applications" $@_{\vdash}$, left/right "projections" $@_1^{\natural}$, $@_2^{\natural}$ and "instantiations" $@_{\cup}$, then we have the following *derived rules*:

- $(\to e)$ $\Gamma\Gamma_1\Gamma_2 \vdash \bigwedge_{\vdash}(\mathrm{y{:}B}).\mathrm{c}[\![\mathrm{y}]\!] \diamondsuit \mathrm{f(a)} : \mathrm{C}$, if $\Gamma[\mathrm{y{:}B}] \vdash \mathrm{c}[\![\mathrm{y}]\!] : \mathrm{C}$, $\Gamma_1 \vdash \mathrm{f} : \mathrm{A} \to \mathrm{B}$, $\Gamma_2 \vdash \mathrm{a} : \mathrm{A}$,
- $(\wedge e)$ $\Gamma\Gamma_1 \vdash \bigwedge_{\natural}(\mathrm{x{:}A,y{:}B}).\mathrm{c}[\![\mathrm{x,y}]\!] \diamondsuit \mathrm{f} : \mathrm{C}$, if $\Gamma[\mathrm{x{:}A}][\mathrm{y{:}B}] \vdash \mathrm{c}[\![\mathrm{x,y}]\!] : \mathrm{C}$, $\Gamma_1 \vdash \mathrm{f} : \mathrm{A} \wedge \mathrm{B}$,
- $(\forall e)$ $\Gamma_u\Gamma\Gamma_1 \vdash \bigwedge_{\cup}(\mathrm{x{:}A}[\![\mathbf{t}]\!]).\mathrm{c}[\![\mathrm{x}]\!] \diamondsuit \mathrm{f}[\mathbf{t}] : \mathrm{C}$, if $\Gamma_u\Gamma[\mathrm{x{:}A}[\![\mathbf{t}]\!]] \vdash \mathrm{c}[\![\mathrm{x}]\!] : \mathrm{C}$, $[\Gamma_u \Vdash \mathbf{t} :: \mathbf{U}]$, $\Gamma_1 \vdash \mathrm{f} : \forall \mathrm{u.A}[\![\mathrm{u}]\!]$.

Conversely, with proof-primitives $\bigwedge_{\vdash}$, $\bigwedge_{\natural}$, and $\bigwedge_{\cup}$ subjected to stratification rules $(\to e)$, $(\wedge e)$, and $(\forall e)$ resp., one can define the *standard "application forms"* f(a), $\mathbf{p}_1(\mathrm{f{:}A{\wedge}B})$, $\mathbf{p}_2(\mathrm{f{:}A{\wedge}B})$ and $\mathrm{f}[\mathbf{t}]$ as above, whereupon the rules $(\to e@_{\vdash})$, $(\wedge e@_1^{\natural})$, $(\wedge e@_2^{\natural})$, and $(\forall i@_{\cup})$ can be *derived* easily, using $< I >$.

Remark (*The negative Boolean* $[\vee,\exists]$*-proof-operators*). With $\vee$, $\exists$ and the associated negative proof-operators $\mathbf{j}$, $\bigvee_{\natural}$ and $\mathbf{J}$, $\bigvee_{\cup}$, resp. defined as above, the following rules are *derivable*:

- $(\vee i)$ $\Gamma \vdash \mathbf{j}(\mathrm{x{:}\neg A,y{:}\neg B}).\mathrm{e}[\![\mathrm{x,y}]\!] : \mathrm{A} \vee \mathrm{B}$, if $\Gamma[\mathrm{x{:}\neg A}][\mathrm{y{:}\neg B}] \vdash \mathrm{e}[\![\mathrm{x,y}]\!] : \bot$,
- $(\vee e)$ $\Gamma\Gamma_1\Gamma_2 \vdash \bigvee_{\natural}(\mathrm{z{:}\neg C}).\mathrm{f} \diamondsuit [\lambda \mathrm{x{:}A.e_1}[\![\mathrm{x,z}]\!], \lambda \mathrm{y{:}B.e_2}[\![\mathrm{y,z}]\!]] : \mathrm{C}$,
  if $\Gamma \vdash \mathrm{f} : \mathrm{A} \vee \mathrm{B}$, $\Gamma_1[\mathrm{x{:}A}][\mathrm{z{:}\neg C}] \vdash \mathrm{e_1}[\![\mathrm{x,z}]\!] : \bot$, $\Gamma_2[\mathrm{y{:}B}][\mathrm{z{:}\neg C}] \vdash \mathrm{e_2}[\![\mathrm{y,z}]\!] : \bot$,
- $(\exists i)$ $\Gamma_u\Gamma \vdash \mathbf{J}(\mathrm{x{:}\neg A}[\![\mathbf{t}]\!]).\mathrm{e}[\![\mathrm{x}]\!] : \exists \mathrm{u.A}[\![\mathrm{u}]\!]$, if $\Gamma_u\Gamma[\mathrm{x{:}\neg A}[\![\mathbf{t}]\!]] \vdash \mathrm{e}[\![\mathrm{x}]\!] : \bot$, $[\Gamma_u \Vdash \mathbf{t} :: \mathbf{U}]$,
- $(\exists e)$ $\Gamma\Gamma_1 \vdash \bigvee_{\cup}(\mathrm{z{:}\neg C}).\mathrm{f} \diamondsuit [!\mathrm{u}.\lambda \mathrm{x{:}A}[\![\mathrm{u}]\!].\mathrm{e}[\![\mathrm{u,x,z}]\!]] : \mathrm{C}$,
  if $\Gamma \vdash \mathrm{f} : \exists \mathrm{u.A}[\![\mathrm{u}]\!]$, $\Gamma_1[\mathrm{u{:}\mathbf{U}}][\mathrm{x{:}A}[\![\mathrm{u}]\!]][\mathrm{z{:}\neg C}] \vdash \mathrm{e}[\![\mathrm{u,x,z}]\!] : \bot$, $[\mathrm{u} \notin \mathrm{FV}_u(\mathrm{C})]$.

Remark (*L-style "left-introductions"*: **CQ**-*"Gentzenization"*).

(1) Using the "initialization" rule $< I >$, the "e-rules" $(\to e)$, $(\wedge e)$, $(\vee e)$, $(\forall e)$ and $(\exists e)$ above can be also instantiated resp. to corresponding rules of "introduction on the left", as in so-called "sequent" (*L*-style) proof-systems:

- $(\to\vdash)$ $\Gamma[\mathrm{f{:}(A{\to}B)}]\Gamma_1 \vdash \bigwedge_{\vdash}(\mathrm{y{:}B}).\mathrm{c}[\![\mathrm{y}]\!] \diamondsuit \mathrm{f(a)} : \mathrm{C}$, if $\Gamma[\mathrm{y{:}B}] \vdash \mathrm{c}[\![\mathrm{y}]\!] : \mathrm{C}$, $\Gamma_1 \vdash \mathrm{a} : \mathrm{A}$,
- $(\wedge\vdash)$ $\Gamma[\mathrm{f{:}(A{\wedge}B)}] \vdash \bigwedge_{\natural}(\mathrm{x{:}A,y{:}B}).\mathrm{c}[\![\mathrm{x,y}]\!] \diamondsuit \mathrm{f} : \mathrm{C}$, if $\Gamma[\mathrm{x{:}A}][\mathrm{y{:}B}] \vdash \mathrm{c}[\![\mathrm{x,y}]\!] : \mathrm{C}$,
- $(\vee\vdash)$ $\Gamma_1\Gamma_2[\mathrm{f{:}(A{\vee}B)}] \vdash \bigvee_{\natural}(\mathrm{z{:}\neg C}).\mathrm{f} \diamondsuit [\lambda \mathrm{x{:}A.e_1}, \lambda \mathrm{y{:}B.e_2}] : \mathrm{C}$,
  if $\Gamma_1[\mathrm{x{:}A}][\mathrm{z{:}\neg C}] \vdash \mathrm{e_1}[\![\mathrm{x,z}]\!] : \bot$, $\Gamma_2[\mathrm{y{:}B}][\mathrm{z{:}\neg C}] \vdash \mathrm{e_2}[\![\mathrm{y,z}]\!] : \bot$
- $(\forall\vdash)$ $\Gamma_u\Gamma[\mathrm{f{:}(\forall u.A}[\![\mathrm{u}]\!])] \vdash \bigwedge_{\cup}(\mathrm{x{:}A}[\![\mathbf{t}]\!]).\mathrm{c}[\![\mathrm{x}]\!] \diamondsuit \mathrm{f}[\mathbf{t}] : \mathrm{C}$, if $\Gamma_u\Gamma[\mathrm{x{:}A}[\![\mathbf{t}]\!]] \vdash \mathrm{c}[\![\mathrm{x}]\!] : \mathrm{C}$, $[\Gamma_u \Vdash \mathbf{t} :: \mathbf{U}]$,
- $(\exists\vdash)$ $\Gamma[\mathrm{f{:}(\exists u.A}[\![\mathrm{u}]\!])] \vdash \bigvee_{\cup}(\mathrm{z{:}\neg C}).\mathrm{f} \diamondsuit [!\mathrm{u}.\lambda \mathrm{x{:}A}[\![\mathrm{u}]\!].\mathrm{e}] : \mathrm{C}$,
  if $\Gamma[\mathrm{u{:}\mathbf{U}}][\mathrm{x{:}A}[\![\mathrm{u}]\!]][\mathrm{z{:}\neg C}] \vdash \mathrm{e}[\![\mathrm{u,x,z}]\!] : \bot$, $[\mathrm{u} \notin \mathrm{FV}_u(\mathrm{C})]$,

where the p-variable f is *fresh* for $(\Gamma_u)\Gamma$, $\Gamma\Gamma_1$, $\Gamma_1\Gamma_2$ resp. Clearly, given the appropriate "cut"-rule(s), the $(\ldots\vdash)$-rules are equivalent to the "e-rules" displayed earlier.

(2) The previous "i-rules", $(\to i\gamma)$ excepted, match analogous rules of "introduction on the right" in "sequent" proof-systems. In other words, the above should also yield a rigorous (*sc.*, *non-elliptic*) *L*-like system for **CQ**, with only "sequents" that are "singular on the right", as in the case of the familiar *L*-systems for **MQ** and/or **HQ**. Here, this is just a notational accident and we don't have any interesting (theoretical) use for it. In particular, the attempt to show that the "cuts" $< \$ >$ and $< \$_{[u]} >$ are *admissible* for an *L*-style **CQ**-stratification that does not have them as primitives is by the way, in the present setting. [The proper theoretical matters to be discussed at this point fall under the rubric *proof-reduction*, which is outside the scope of this work.]

Remarks (*Alternative Boolean proof-term stratifications*).

(1) The *extended proof-term stratification* $\vdash_*[\mathbf{CQ}]$ is supposed to be defined by:

- *proof-context rules*:
  - *"structural"*: $< I >$, $< K >$, $< K_u >$,
  - *"cuts"*: $< \$ >$, $< \$_{[u]} >$, (possibly: $< \$K >$, $< \$_u K >$),
- *"type-assignment" rules*:
  - *inferential*: $(\rightarrow i\lambda)$, $(\rightarrow i\gamma)$, $(\rightarrow e\mathbf{@}_{\vdash})$,
  - *algebraic*: $(\wedge i)$, $(\wedge e\mathbf{@}^{\natural}_1)$, $(\wedge e\mathbf{@}^{\natural}_2)$, and $(\vee i)$, $(\vee e)$,
  - *generic* [*first-order*]: $(\forall i)$, $(\forall e\mathbf{@}_{\cup})$, and $(\exists i)$, $(\exists e)$.

If the proof-contexts are thought of as sequences, one could take as primitive proof-context rules

- *the "structural" rules*:
  - $< I >$, $< K >$, $< KW >$, $< W >$, $< C >$,
  - $< K_u >$, $< KW_u >$, $< W_u >$, $< C_u >$, and
- *the "cuts"*:
  - $< \$K >$, $< \$W >$, $< \$ >$,
  - $< \$_u K >$, $< \$_u W >$, $< \$_{[u]} >$.

As mentioned earlier, this set is slightly redundant for the purposes of **CQ**.

(2) *The standard stratifications* $\vdash_{\&}[\mathbf{CQ}]$ *and* $\vdash[\mathbf{CQ}]$. So $\vdash_{\&}[\mathbf{CQ}]$ is the restriction of $\vdash_*[\mathbf{CQ}]$ to a $[\bot,\rightarrow,\wedge,\forall]$ type-structure, i.e., to a proof-language without $[\vee,\exists]$-proof-primitives [missing rules: $(\vee i)$, $(\vee e)$, and $(\exists i)$, $(\exists e)$]. Also, $\vdash[\mathbf{CQ}]$ is the restriction of $\vdash_{\&}[\mathbf{CQ}]$ to $[\bot,\rightarrow,\forall]$ (no algebraic proof-primitives). The stratification $\vdash[\mathbf{CQ}]$ makes up *a minimal syntactic setting* for a **CQ**-proof-theory (in the *technical sense* of this paper).

(3) The proof-term-structures $\vdash_*[\mathbf{CQ}]$, and $\vdash_{\&}[\mathbf{CQ}]$ are *stratification-equivalent*. Specifically, if the Boolean $[\vee,\exists]$-operators $\mathbf{j}$, $\mathbf{J}$, and $\bigvee_{\natural}$, $\bigvee_{\cup}$ are defined as above, let $[\mathrm{a}]_*$ be the definitional expansion of a in $\vdash_*[\mathbf{CQ}]$ (relative to these definitions), for all proof-terms a in $\vdash_*[\mathbf{CQ}]$. Then, where $\vdash_*$ and $\vdash_{\&}$ resp. have the expected meaning, we have

$$\Gamma \vdash_{\&} [\mathrm{a}]_* : \mathrm{A} \iff \Gamma \vdash_* \mathrm{a} : \mathrm{A}, \text{ for all p-terms a of } \vdash_*[\mathbf{CQ}].$$

Indeed, since $[\mathrm{a}]_* \equiv \mathrm{a}$ for the proof-terms of $\vdash_{\&}[\mathbf{CQ}]$, the $\Rightarrow$-part of the statement is immediate. The converse presupposes the fact that we are able to derive the algebraic rules $(\vee i)$, $(\vee e)$, and the first-order rules $(\exists i)$ and $(\exists e)$ within $\vdash_{\&}[\mathbf{CQ}]$, using the previous definitions of the Boolean $[\vee,\exists]$-proof-operators $\mathbf{j}$, $\mathbf{J}$, $\bigvee_{\natural}$, $\bigvee_{\cup}$. This is straightforward. So, for classical logic purposes, there is no loss of generality if we restrict the considerations to the proof-terms of $\vdash_{\&}[\mathbf{CQ}]$, based on $[\bot,\rightarrow,\wedge,\forall]$.

(4) *Intensional proof-theories for* **CQ**. One can show that $\vdash_{\&}[\mathbf{CQ}]$ and $\vdash[\mathbf{CQ}]$ are stratification-equivalent, too, although the latter fact has no direct equational counterpart (reason: as already mentioned above, the proof-theories based on $\vdash_{\&}[\mathbf{CQ}]$ have also "$\wedge$-extensionality" $[\wedge\eta]$ [i.e., "surjectivity of pairing"] as a postulate, and the latter cannot be simulated in inferential terms). So, the $\lambda\gamma$-calculi based on $\vdash[\mathbf{CQ}]$ alone are "intensional" in the sense that one cannot associate extensional proof-operations to conjunctions (and/or disjunctions) that are available definitionally in terms of a $[\bot,\rightarrow,(\forall)]$-type-structure. [In fact, any definition $\varphi[\mathrm{A,B}]$ of $(\mathrm{A} \wedge \mathrm{B})$ or $(\mathrm{A} \vee \mathrm{B})$ in terms of $[\bot,\rightarrow,(\forall)]$, such that A, B occur in "negative" sub-formulas in $\varphi[\mathrm{A,B}]$ would admit of "intensional" proof-operators that are satisfying the appropriate β-rules, but fail to satisfy analogous η-conditions. These derived operators are, in general, *distinct from each other*.]

Provability-completeness for $\vdash_*[\mathbf{CQ}]$, $\vdash_{\&}[\mathbf{CQ}]$, *etc.* It is easy to establish the fact that $\mathbf{CQ} \Vdash \mathrm{A} \iff [\,] \vdash_* \mathrm{a} : \mathrm{A}$, for some Boolean proof-combinator a of $\vdash_*[\mathbf{CQ}]$ (where "$\mathbf{CQ} \Vdash \mathrm{A}$" reads "A is provable in **CQ**"). Analogous statements hold for $\vdash_{\&}[\mathbf{CQ}]$ and $\vdash[\mathbf{CQ}]$. This must be *intuitively clear* by comparing the

$\vdash_*$[**CQ**]-rules above with the rules of an appropriate N-formulation for **CQ** (e.g., [Prawitz 65].) In view of the above, it is enough to show this for $\vdash$[**CQ**].[20]

The Minimalkalkül as a typed λ-calculus. We can examine now the sub-structure, $\vdash$[**MQ**] say, of $\vdash_*$[**CQ**] corresponding to Johansson's *Minimalkalkül* **MQ**. Recall that, on a (syntactic) proof-term level, the "positive"/"minimal" instances of the "negative" sumptors (the "Boolean injections") **j**, **J** were given by proof-operators $\mathbf{j}_i$, [i := 1,2], and $\mathbf{J}_m$, while the "positive"/"minimal" contents of the Boolean negative [$\vee$,$\exists$]-selectors $\bigvee_\natural$, $\bigvee_\cup$ were recorded by the "minimal selectors" $\sqcup$, $\amalg$, resp.

Explicitly, with Boolean $\vee$-proof-primitives, one has, in $\vdash_*$[**CQ**] (w.r.t. the [$\bot$,$\to$,$\wedge$,$\vee$,$\forall$,$\exists$] type-structure), the following analogues of the *Minimalkalkül*-stratification ("type-assignment") rules for the "positive" ["minimal"] (and thus intuitionistic) disjunction $\vee_m$.

Lemma (*Johansson-Heyting $\vee$-rules*: *Minimalkalkül $\vee$-stratification*).

$(\vee i_1)_m$ $\Gamma \vdash \mathbf{j}_1[\![A,B]\!](a{:}A) : A \vee B$, if $\Gamma \vdash a : A$,
$(\vee i_2)_m$ $\Gamma \vdash \mathbf{j}_2[\![A,B]\!](b{:}B) : A \vee B$, if $\Gamma \vdash b : B$,
$(\vee e)_m$ $\Gamma\Gamma_1\Gamma_2 \vdash \sqcup(f,[x{:}A].c_1[\![x]\!]{:}C,[y{:}B].c_2[\![y]\!]{:}C) : C$,
if $\Gamma \vdash f : A \vee B$, $\Gamma_1[x{:}A] \vdash c_1[\![x]\!] : C$, $\Gamma_2[y{:}B] \vdash c_2[\![y]\!] : C$.

Proof. In $\vdash_*$[**C**(**Q**)], i.e., w.r.t. [$\bot$,$\to$,$\wedge$,$\vee$,($\forall$,$\exists$)], one needs, for $(\vee i_i)_m$, [i := 1,2]: <**K**>, and $(\vee i)$, whereas, for $(\vee e)_m$: $< I >$, ($\to e$@$_\vdash$), and $(\vee e)$ suffice. □

Similarly, one obtains in $\vdash_*$[**CQ**], with Boolean $\exists$-proof-primitives, the following analogues of the Heyting/Johansson stratification proof-rules for the "minimal"/"positive" (and thus intuitionistic) existential quantifier $\exists_m$.

Lemma (*Johansson-Heyting $\exists$-rules*: *Minimalkalkül $\exists$-stratification*).

$(\exists i)_m$ $\Gamma_u\Gamma \vdash [\mathbf{t},a{:}A[\![\mathbf{t}]\!]]$ [$\equiv \mathbf{J}_m(a{:}A[\![\mathbf{t}]\!])$] : $\exists u.A[\![u]\!]$, if $\Gamma_u\Gamma \vdash a : A[\![\mathbf{t}]\!]$, [$\Gamma_u \Vdash \mathbf{t} :: \mathbf{U}$],
$(\exists e)_m$ $\Gamma\Gamma_1 \vdash \amalg(f,[u{:}\mathbf{U}][x{:}A[\![u]\!]].c[\![u,x]\!]{:}C) : C$, if $\Gamma \vdash f : \exists u.A[\![u]\!]$, $\Gamma_1[u{:}\mathbf{U}][x{:}A[\![u]\!]] \vdash c[\![u,x]\!] : C$,
[$u \notin FV_u(C)$].

Proof. In $\vdash_*$[**CQ**] (i.e., w.r.t. [$\bot$,$\to$,$\wedge$,$\vee$,$\forall$,$\exists$]), one needs, for $(\exists i)_m$: $< I >$, ($\to e$@$_\vdash$), $(\exists i)$, and, for $(\exists e)_m$: $< I >$, ($\to e$@$_\vdash$), $(\exists e)$. □

The following "positive" β-type [$\vee$,$\exists$]-rules are derivable in $\lambda\gamma_\&$**CQ**.

Lemma (*Minimalkalkül β-[$\vee$,$\exists$]$_m$-proof-behavior in $\lambda\gamma_\&$***CQ**).

(1) *Functional rules.*
(11) *β-$\vee_m$-rules*:
$[\beta\vee_1]_m$ $\Gamma \vdash \sqcup(\mathbf{j}_1[\![A,B]\!](a),[x{:}A].c_1[\![x]\!],[y{:}B].c_2[\![y]\!]) = c_1[\![x{:=}a]\!] : C$,
if $\Gamma \vdash a : A$, $\Gamma[x{:}A] \vdash c_1[\![x]\!] : C$, $\Gamma[y{:}B] \vdash c_2[\![y]\!] : C$,
$[\beta\vee_2]_m$ $\Gamma \vdash \sqcup(\mathbf{j}_2[\![A,B]\!](b),[x{:}A].c_1[\![x]\!],[y{:}B].c_2[\![y]\!]) = c_2[\![y{:=}b]\!] : C$,
if $\Gamma \vdash b : B$, $\Gamma[x{:}A] \vdash c_1[\![x]\!] : C$, $\Gamma[y{:}B] \vdash c_2[\![y]\!] : C$,

(12) *β-$\exists_m$-rule*:
$[\beta\exists]_m$ $\Gamma \vdash \amalg([\mathbf{t},a{:}A[\![\mathbf{t}]\!]],[u{:}\mathbf{U}][x{:}A[\![u]\!]].c[\![u,x]\!]) = c[\![u{:=}\mathbf{t}]\!][\![x{:=}a]\!] : C$,
if $\Gamma \Vdash \mathbf{t} :: \mathbf{U}$, $\Gamma \vdash a : A[\![u{:=}\mathbf{t}]\!]$, $\Gamma[u : \mathbf{U}][x : A[\![u]\!]] \vdash c[\![u,x]\!] : C$, [$u \notin FV_u(C)$].

(2) *Congruence rules for $\vee_m$ and $\exists_m$.*

[20] If the "purity of methods" is at premium, one could look up [Rezuş 90], for *a technical alternative*: it consists of using a *combinatory stratification* equivalent of $\vdash$[**CQ**], as an intermediate step. Thereby the problem is reduced to a matter of *evidence*, too, the issue being in the fact that one should rather trust the *axiomatics*, i.e., a [finite] set of *witness-patterns*. A *clean* – although not very deep – way of obtaining this type of – otherwise obvious – result would be, likely, by constructing a model for classical provability, just from Boolean combinators [*exercise*].

$[\xi\vee_i]_m$ $\Gamma \vdash \mathbf{j}_i[\![A_1,A_2]\!](a_1{:}A_i) = \mathbf{j}_i[\![A_1,A_2]\!](a_2{:}A_i) : A_1 \vee A_2$, if $\Gamma \vdash a_1 = a_2 : A_i$, [i := 1,2],
$[\mu\vee]_m$ $\Gamma \vdash \sqcup(f_1,[x{:}A].a_1[\![x]\!],[y{:}B].b_1[\![y]\!]) = \sqcup(f_2,[x{:}A].a_2[\![x]\!],[y{:}B].b_2[\![y]\!]) : C$,
if $\Gamma[x{:}A] \vdash a_1[\![x]\!] = a_2[\![x]\!] : C$, $\Gamma \vdash f_1 = f_2 : A \vee B$, $\Gamma[y{:}B] \vdash b_1[\![y]\!] = b_2[\![y]\!] : C$,
$[\xi\exists]_m$ $\Gamma \vdash [\ \mathbf{t}, a_1{:}A[\![\mathbf{t}]\!]\ ] = [\ \mathbf{t}, a_2{:}A[\![\mathbf{t}]\!]\ ] : \exists u.A[\![u]\!]$, if $\Gamma \vdash a_1 = a_2 : A[\![u{:=}\mathbf{t}]\!]$, $[\Gamma \Vdash \mathbf{t} :: \mathbf{U}]$,
$[\mu\exists]_m$ $\Gamma \vdash \amalg(f_1,[u{:}\mathbf{U}][x{:}A[\![u]\!]].c_1[\![u,x]\!]) = \amalg(f_2,[u{:}\mathbf{U}][x{:}A[\![u]\!]].c_2[\![u,x]\!]) : C$,
if $\Gamma \vdash f_1 = f_2$: $\exists u.A[\![u]\!]$, $\Gamma[u : \mathbf{U}][x : A[\![u]\!]] \vdash c_1[\![u,x]\!] = c_2[\![u,x]\!] : C$, $[u \notin FV_u(C)]$.

Proof. Straightforward calculations. The *explicit* derivations in $\lambda\gamma_{\&}\mathbf{CQ}$ are informative, however [*exercise*]. We list only the required derivability conditions (ignoring congruence rules). For $[\beta\vee_1]_m$: $[\beta \to \lambda]$, $[\beta\wedge_1]$, and $[\eta \to \gamma]$. For $[\beta\vee_2]_m$: Analogously. For $[\beta\exists]_m$: $[\beta \to \lambda]$, $[\beta\forall]$, and $[\eta \to \gamma]$. For $[\xi\vee_1]_m$, $[\xi\vee_2]_m$, $[\mu\vee]_m$, $[\xi\exists]_m$, $[\mu\exists]_m$: use the primitive $\lambda\gamma_{\&}\mathbf{CQ}$-congruence rules. □

The "extensionality" [∨,∃]-rules of $\lambda\gamma_{\&}\mathbf{CQ}$ are already "minimal", viz., using the (simulated) **MQ**-notation, one has, in $\lambda\gamma_{\&}\mathbf{CQ}$:

Lemma (*Boolean* $[\vee,\exists]_m$*-"extensionality" in Minimalkalkül notation*).

$[\eta\vee]_m$ $\Gamma \vdash \sqcup(f,[x{:}A].\mathbf{j}_1[\![A,B]\!](x),[y{:}B].\mathbf{j}_2[\![A,B]\!](y)) = f$, if $\Gamma \vdash f : A \vee B$, $[x, y \notin FV_\lambda(f)]$,
$[\eta\exists]_m$ $\Gamma \vdash \amalg(f,[u{:}\mathbf{U}][x{:}A[\![u]\!]].[u,x{:}A[\![u]\!]]) = f$, if $\Gamma \vdash f : \exists u.A[\![u]\!]$, $[x \notin FV_\lambda(f), u \notin FV_u(f)]$.

Proof. Straightforward [*exercise*]. The *explicit* derivability conditions in $\lambda\gamma_{\&}\mathbf{CQ}$ are, for $[\eta\vee]_m$: $[\beta \to \lambda]$, $[\eta \to \lambda]$, $[\eta\wedge]$, $[\beta\gamma\bot]$, and $[{}^h\beta\gamma \to]$ and, for $[\eta\exists]_m$: $[\beta \to \lambda]$, $[\eta \to \lambda]$, $[\eta\forall]$, $[\beta\gamma\bot]$, and $[{}^h\beta\gamma \to]$. □

Remark ($\lambda\gamma_{\&}\mathbf{CQ}$*-derivability conditions for the Minimalkalkül*). Inspecting the *explicit* derivations above shows that neither $[\oint\gamma]$ nor the "[∧,∀]-extensionality" properties of $\lambda\gamma_{\&}\mathbf{CQ}$ were needed in the derivation of the "positive" β-rules and in the $[\vee,\exists]_m$-congruence rules. So the "positive" β-rules $[\beta\vee_1]_m$, $[\beta\vee_2]_m$, $[\beta\exists]_m$ resp., as well as the congruence rules $[\xi\vee_1]_m$, $[\xi\vee_2]_m$, $[\mu\vee]_m$ and $[\xi\exists]_m$, $[\mu\exists]_m$ resp. are derivable in the diagonalization-free subsystem of $\lambda\gamma_{\&}\mathbf{CQ}$. [For the record, however, "γ-extensionality" $[\eta \to \gamma]$ is apparently unavoidable in the derivation of the "positive" β-$[\vee,\exists]_m$-rules.] On the other hand, from the *explicit* derivation of $[\eta\vee]_m$, it is clear that diagonalization $[\oint\gamma]$ is *not* needed in order to get $[\eta\vee]_m$ in $\lambda\gamma_{\&}\mathbf{CQ}$ either, although $[\eta\wedge]$ *is* required. Thus $[\eta\vee]_m$ ("∨-extensionality") obtains in a *proper* subsystem of $\lambda\gamma_{\&}\mathbf{CQ}$ *without* diagonalization, but *with* "∧-extensionality" ("surjectivity of pairing"). Actually, the property $[\eta\vee]_m$ is rarely, if ever, mentioned in the literature on **MQ** and the Heyting logic. Finally, inspecting the *explicit* derivation of $[\eta\exists]_m$ shows that the "∃-extensionality" property obtains already in a subsystem of $\lambda\gamma_{\&}\mathbf{CQ}$ without diagonalization, although one needs $[\eta\forall]$ ("∀-extensionality"). As suggested above, the "positive extensionalities" $[\eta\vee]_m$ and $[\eta\exists]_m$ are - very likely - *the most general* [∨,∃]*-extensionality conditions derivable in* $\lambda\gamma_{\&}\mathbf{CQ}$.

One can now isolate the proof-theory of the *Minimalkalkül* as a *typed* λ*-calculus* $\lambda\mathbf{MQ}$. This is a proper extension of $\lambda\pi!$, too.

We define the proof-syntax of **MQ** and the stratification $\vdash$[**MQ**] of **MQ**-proof-terms (w.r.t. the full [⊥,→,∧,∨,∀,∃] type-structure), as a fragment of $\vdash_*$[**CQ**], *modulo* the obvious definitional embedding into $\vdash_{\&}$[**CQ**]. In order to do this, one can use the γ-free syntax of $\vdash_*$[**CQ**] with the **CQ**-primitives **j**, **J**, and $\bigvee_\natural$, $\bigvee_\cup$, resp. replaced by appropriate "minimal" primitives $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{J}_m$ and ⊔, ∐, resp. (with primitive proof-forms matching those defined above), subjected to the "minimal" stratification rules $(\vee i_1)_m$, $(\vee i_2)_m$, $(\vee e)_m$, $(\exists i)_m$, $(\exists e)_m$, resp. in place of the $\vdash_*$[**CQ**]-rules $(\vee i)$, $(\vee e)$, $(\exists i)$, and $(\exists e)$, resp. (the remaining items being as in $\vdash_*$[**CQ**]).

The *equational theory* $\lambda\mathbf{MQ}$ – the proof-theory of the *Minimalkalkül* – is obtained in the same way, using, in the case of the "minimal" [∨,∃]-postulates, primitive analogues of the "positive" $\beta\eta$-conditions, shown to be $\lambda\gamma_{\&}\mathbf{CQ}$-derivable in the above.

For further reference, the *Minimalkalkül proof-syntax* – relative to [⊥,→,∧,∨,∀,∃]-types –, the *proof-rules* for the stratification $\vdash$[**MQ**], and the *equational theory* $\lambda\mathbf{MQ}$ are given by the following

Definition (*The Minimalkalkül as a typed* λ*-calculus*: λ**MQ**).

(1) *Minimalkalkül proof-operators*:

- *inferential operators*:
  - *sumptors*: $\lambda_\vdash(\dots)$,
  - *selectors*: $@_\vdash(\dots,\dots)$ [derived: $\bigwedge_\vdash(\dots,\dots,\dots)$],
- *algebraic operators* (i := 1,2):
  - *sumptors*: $\lambda_\natural(\dots,\dots)$, $\mathbf{j}_i(\dots)$,
  - *selectors*: $@_i^\natural(\dots)$, [derived: $\bigwedge_\natural(\dots)$], $\sqcup(\dots,\dots,\dots)$,
- *generic operators*:
  - *sumptors*: $\lambda_\cup(\dots)$, $\mathbf{J}_m(\dots)$
  - *selectors*: $@_\cup(\dots,\dots)$, [derived: $\bigwedge_\cup(\dots)$], II$(\dots,\dots)$.

(2) *Minimalkalkül proof-terms* [primitive forms]:

- *inferential*:

  λx:A.b⟦x⟧ [≡ $\lambda_\vdash$([x:A].b⟦x⟧:B)],
  f(a) [≡ $@_\vdash$(f:A→B,a:A)],

- *algebraic*:

  <a:A,b:B> [≡ $\lambda_\natural$(a:A,b:B)],
  $\mathbf{p}_1$(c:A∧B) [≡ $@_1^\natural$(c:A∧B)],
  $\mathbf{p}_2$(c:A∧B) [≡ $@_2^\natural$(c:A∧B)],
  $\mathbf{j}_1$⟦A,B⟧(a:A) [≡ $\mathbf{j}_1$(a:A,⟦B⟧)],
  $\mathbf{j}_2$⟦A,B⟧(b:B) [≡ $\mathbf{j}_2$(⟦A⟧,b:B)],
  ⊔(f,[x:A].c_1⟦x⟧:C,[y:B].c_2⟦y⟧:C),

- *generic* [*first-order*]:

  !u.a⟦u⟧ [≡ $\lambda_\cup$([u:**U**]a:A⟦u⟧)],
  f[**t**] [≡ $@_\cup$(f:∀u.A⟦u⟧,**t**:**U**)],
  [**t**,a:A⟦u:=**t**⟧] [≡ $\mathbf{J}_m$(x:A⟦u:=**t**⟧)],
  II(f,[u:**U**][x:A⟦u⟧].c⟦u,x⟧:C), [u $\notin$ FV_u(C)].

(3) *Minimalkalkül proof-rules* (*the* ⊢[**MQ**]*-stratification*):

- *proof-context rules* (as for $\vdash_{(\&,\star)}$[**CQ**]):
  - *"structural"*: $< I >$, $< K >$, $< K_u >$,
  - *"cuts"*: $< \$ >$, $< \$_{[u]} >$ (possibly: $< \$K >$, $< \$_u K >$),
- *"type-assignment" rules*:
  - *inferential*: $(\rightarrow i\lambda)$, $(\rightarrow e@_\vdash)$,
  - *algebraic*: $(\wedge i)$, $(\wedge e@_1^\natural)$, $(\wedge e@_2^\natural)$, $(\vee i_1)_m$, $(\vee i_2)_m$, $(\vee e)_m$,
  - *generic* [*first-order*]: $(\forall i)$, $(\forall i@_\cup)$, $(\exists i)_m$, $(\exists e)_m$.

(4) λ**MQ** is an equational theory – a typed λ-theory – defined on ⊢[**MQ**] by the following equational postulates [proper "rules" and congruence conditions]:

- $[\beta\rightarrow\lambda]$, $[\eta\rightarrow\lambda]$, as in $\lambda\gamma_\&$**CQ**,
- $[\beta\wedge_i]$, $[\eta\wedge]$, (i := 1,2), as in $\lambda\gamma_\&$**CQ**,
- $[\beta\vee_i]_m$, $[\eta\vee]_m$, (i := 1,2), specific,
- $[\beta\forall]$, $[\eta\forall]$, as in $\lambda\gamma_\&$**CQ**,
- $[\beta\exists]_m$, $[\eta\exists]_m$, specific,
- $[\xi\rightarrow\lambda]$, $[\mu\rightarrow]$, $[\nu\rightarrow]$, as in $\lambda\gamma_\&$**CQ**,
- $[\xi\wedge]$, $[\nu\wedge_i]$, (i := 1,2), as in $\lambda\gamma_\&$**CQ**,

- $[\xi\vee_i]_m$, $[\mu\vee]_m$, (i := 1,2), specific,
- $[\xi\forall]$, $[\mu\forall]$, as in $\lambda\gamma_{\&}$**CQ**,
- $[\xi\exists]_m$, $[\mu\exists]_m$, specific,

where the $[\vee,\exists]_m$-rules are, *mutatis mutandis*, as above.

Alternatively, in place of the primitive "application forms" $@_{\vdash}$, $@_1^{\natural}$, $@_2^{\natural}$ and $@_{\cup}$, one could have taken as primitives the general Boolean *positive selectors*. These are definable in *Minimalkalkül*, exactly as in the Boolean case.

Remark (λ**MQ**, *the Heyting calculus and Martin-Löf's type theory*).

(1) $\vdash$[**MQ**] is a proper fragment of the *Heyting proof-syntax* and the *Heyting proof-term stratification* $\vdash$[**HQ**], discussed below, (i.e., the **MQ**-operators $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{J}_m$, $\sqcup$ and $\amalg$ are also proof-operators of the Heyting first-order proof-calculus). Indeed, the $[\vee,\exists]_m$-rules above are better-known as **HQ**-rules. Notably, $\vdash$[**HQ**] has *more* proof-forms (essentially, the ω-terms); these require, of course, *additional* stratification rules.

(2) Less emphasized in current discussions of "constructivism" is the fact that the so-called "Heyting proof-calculus" λ**HQ** is a typed λ-calculus (= an equational theory) extending *properly* λ**MQ**, even *in the* $\vdash$[**MQ**]-*syntax*, viz. there are proof-equations of λ**HQ** (as, e.g., the "commuting conversions") that are not derivable in λ**MQ**, but can be already written down with the "purely positive" expressive means of $\vdash$[**MQ**]. In other words, λ**MQ** is a *proper* fragment of the ω-free part of the (full) Heyting (first-order) proof-calculus.

(3) In particular, λ**MQ** (as an equational theory) is, essentially, the same thing as the *first-order fragment* of Martin-Löf's [84] constructive type theory **CTT**. [The current type-theoretic literature is rather loose about the *proof-theoretic* import of Martin-Löf's type theory.] Of course, since the "empty type" $\perp$ (Martin-Löf's $\mathbf{N}_0$), and an analogue of our ω are also present in the primitive syntax of **CTT**, one has additional congruence ω-conditions in (the first-order) **CTT**, with *no further effect*, however. Like in λ**MQ**, none of the Heyting "$\perp$-rules" (here: ω-rules) are derivable in this fragment of **CTT**, let alone the "Heyting commuting conversions" of λ**HQ**. The additional proof-theoretic (logical) strength of Martin-Löf's system(s) is obtained by generalizing the quantifier rules "at a higher-order level", using an *ad hoc* hierarchy of *universes*, subjected to appropriate closure conditions. But this does *not* help us retrieve even the bare "0-order" (propositional) fragment of the full Heyting proof-calculus. In fact, this situation is *as* [originally] *intended*, since Martin-Löf's type theory has been meant to provide a formal counterpart for Bishop's constructive mathematics (**BCM**). Reputedly, the latter can be also viewed as a formidable *tour de force* in the attempt of reconstructing the bulk of current mathematics in "pure positive terms". Brouwer's intuitionism (**BI**) is less restrictive as regards the use of the so-called "negative properties". This is also reflected in the fact that the [proof-theory of the] Heyting logic is significantly more complex than that [of the logic] presupposed by **BCM**.

From the above, we retain the fact that $\lambda\gamma_{\&}$**CQ** contains equationally the proof-theory of *Minimalkalkül* (λ**MQ**), as a *proper extension*, *modulo* an appropriate *definitional embedding* $[\ldots]_{\&} : \lambda\mathbf{MQ} \longrightarrow \lambda\gamma_{\&}\mathbf{CQ}$, say. [The extension is *proper*, since λ**MQ** is a γ-free system.]

For the record, we collect the information of this section, in a

Theorem ($\lambda\gamma_{\&}\mathbf{CQ} \supset [\lambda\mathbf{MQ}]_{\&}$). $\lambda\gamma_{\&}$**CQ** is a proper equational extension of λ**MQ**, the proof-theory of *Minimalkalkül* (in the *definitional embedding* sense).

Proof. Completed in the above. □

This yields, of course, **Cons**(λ**MQ**), i.e., Post-consistecy for λ**MQ**, well-known also by different means (the proof-theory of **HQ** [Prawitz 65], Martin-Löf's **CTT**, the *Automath* literature, etc.).

A less abrupt generalization of the $[\vee,\exists]_m$-proof-operators than that available in **CQ** can be obtained from the following *extended exercise*, in two parts [part two comes at the end of the next section].

Exercise (*The "negative"* **DQ**-*proof operators*: *first part*). State and derive, in $\lambda\gamma_{\&}\mathbf{CQ}$, the **DQ**-analogues ($[\vee,\exists]_d$-rules) of the *Minimalkalkül* $[\vee,\exists]$-rules (using, to this effect, the Boolean simulation of the "Clavian" proof-operators $\mathbf{j}^1$, $\mathbf{j}^2$, $\mathbf{J}_d$, $\sqcup_d$, $\amalg_d$, suggested earlier *sub* PROOF-SYNTAX).

Chapter V

Negative proof-operators: injections and [∨,∃]-selectors

We consider next the full structure $\vdash_*[\mathbf{CQ}]$ and $\lambda\gamma$-theories, based on [⊥,→,∧,∨,∀,∃]. Specifically, we investigate the *most general* equational behavior of the *negative* Boolean *proof operations* associated to (the classical) [∨,∃], as available, ultimately, under the stratification *régime* of $\vdash_*[\mathbf{CQ}]$.

One could define an equational system $\lambda\gamma_{[*]}\mathbf{CQ}$ on $\vdash_*[\mathbf{CQ}]$ that extends $\lambda\gamma_{\&}\mathbf{CQ}$ properly in the $\vdash_*[\mathbf{CQ}]$-syntax. Using the standard λ-calculus terminology, this is a (*typed*) $\lambda\gamma$-*theory*. From an intuitive point of view, $\lambda\gamma_{[*]}\mathbf{CQ}$ must be identified with *the proof-theory of first-order classical logic based on* [⊥,→,∧,∨,∀,∃]. $\lambda\gamma_{[*]}\mathbf{CQ}$ can be formulated explicitly, by adding (a rather long list of rules as) equational postulates to $\lambda\gamma_{\&}\mathbf{CQ}$: one can obtain these postulates first as derived rules [equations] of $\lambda\gamma_{\&}\mathbf{CQ}$). For present purposes, we don't need the full $\lambda\gamma_{[*]}\mathbf{CQ}$. Mentioned next are only those rules of $\lambda\gamma_{[*]}\mathbf{CQ}$ that have been estimated relevant in the discussion of the first-order Heyting proof-calculus.

Assumed in what follows are the standard Boolean definitions of ∨, ∃ and the canonical simulation of the [∨,∃] proof-operations of $\vdash_*[\mathbf{CQ}]$ in terms of "positive" proof-operations and γ-abstraction.

[$\beta\eta$-∨]-*and* [$\beta\eta$-∃]-*rules in* $\lambda\gamma_{\&}\mathbf{CQ}$. We obtain first the most general [∨,∃]-analogues of the ($\lambda\gamma$)-$\beta\eta$-postulates of $\lambda\gamma_{\&}\mathbf{CQ}$.

Heuristically, we have to identify the [∨,∃]-proof-*détours*[21] as well as the "complex" applications of *reductio ad absurdum*, beyond the [⊥,→,∧,∀]-proof-syntax (i.e., *reductio* from [∨,∃]-formulas). The latter should yield the reduction/equational behavior of p-terms γx:¬A.e⟦x⟧, for A ≡ [B ∨ C] and A ≡ [∃u.B⟦u⟧].

The following $\beta\eta$-[∨,∃]-rules are derivable in $\lambda\gamma_{\&}\mathbf{CQ}$, *modulo* the canonical definition of the [∨,∃]-proof-operations in $\vdash_{\&}[\mathbf{CQ}]$. [In order to ease readability, the official *Minimalkalkül*-notation is used, whenever this is possible.]

Theorem ($\beta\eta$-[∨,∃]-*rules in* $\lambda\gamma_{\&}\mathbf{CQ}$).

(1) *Evaluation rules*.

(111) β∨-*evaluation*:

[β∨] $\Gamma \vdash \bigvee_{\natural}(\mathrm{z}{:}\neg\mathrm{C}).(\mathbf{j}(\mathrm{x}{:}\neg\mathrm{A},\mathrm{y}{:}\neg\mathrm{B}).\mathrm{e}[\![\mathrm{x},\mathrm{y}]\!]) \diamondsuit [\lambda\mathrm{x}_0{:}\mathrm{A}.\mathrm{e}_1,\lambda\mathrm{y}_0{:}\mathrm{B}.\mathrm{e}_2] = \gamma\mathrm{z}{:}\neg\mathrm{C}.\mathrm{e}[\![\mathrm{x}{:=}\mathrm{c}_1]\!][\![\mathrm{y}{:=}\mathrm{c}_2]\!]\ [:\ \mathrm{C}]$,
if $\Gamma[\mathrm{x}{:}\neg\mathrm{A}][\mathrm{y}{:}\neg\mathrm{B}] \vdash \mathrm{e}[\![\mathrm{x},\mathrm{y}]\!] : \bot$, $\Gamma[\mathrm{x}_0{:}\mathrm{A}][\mathrm{z}{:}\neg\mathrm{C}] \vdash \mathrm{e}_1[\![\mathrm{x}_0,\mathrm{z}]\!] : \bot$, $\Gamma[\mathrm{y}_0{:}\mathrm{B}][\mathrm{z}{:}\neg\mathrm{C}] \vdash \mathrm{e}_2[\![\mathrm{y}_0,\mathrm{z}]\!] : \bot$,
where $\mathrm{c}_1[\![\mathrm{z}]\!] \equiv \lambda\mathrm{x}_0{:}\mathrm{A}.\mathrm{e}_1[\![\mathrm{x}_0,\mathrm{z}]\!]$ is free for x in $\mathrm{e}[\![\mathrm{x},\mathrm{y}]\!]$, and $\mathrm{c}_2[\![\mathrm{z}]\!] \equiv \lambda\mathrm{y}_0{:}\mathrm{B}.\mathrm{e}_2[\![\mathrm{y}_0,\mathrm{z}]\!]$ is free for y in $\mathrm{e}[\![\mathrm{x},\mathrm{y}]\!]$,

(112) "∨-*extensionality*":

[η∨] $\Gamma \vdash \bigvee_{\natural}(\mathrm{z}{:}\neg(\mathrm{A}\vee\mathrm{B})).\mathrm{f} \diamondsuit [\lambda\mathrm{x}{:}\mathrm{A}.\mathrm{z}(\mathrm{e}_1[\![\mathrm{x}]\!]),\lambda\mathrm{y}{:}\mathrm{B}.\mathrm{z}(\mathrm{e}_2[\![\mathrm{y}]\!])] = \mathrm{f}$, if $\Gamma \vdash \mathrm{f} : \mathrm{A} \vee \mathrm{B}$,
where $\mathrm{e}_1[\![\mathrm{x}]\!] \equiv \mathbf{j}_1[\![\mathrm{A},\mathrm{B}]\!](\mathrm{x}{:}\mathrm{A}) \equiv \mathbf{j}(\mathrm{x}_1{:}\neg\mathrm{A},\mathrm{y}_1{:}\neg\mathrm{B}).\mathrm{x}_1(\mathrm{x})$, $\mathrm{e}_2[\![\mathrm{y}]\!] \equiv \mathbf{j}_2[\![\mathrm{A},\mathrm{B}]\!](\mathrm{y}{:}\mathrm{B}) \equiv \mathbf{j}(\mathrm{x}_1{:}\neg\mathrm{A},\mathrm{y}_1{:}\neg\mathrm{B}).\mathrm{y}_1(\mathrm{y})$,

(121) β∃-*evaluation*:

[β∃] $\Gamma \vdash \bigvee_{\cup}(\mathrm{z}{:}\neg\mathrm{C}).(\mathbf{J}(\mathrm{x}{:}\neg\mathrm{A}[\![\mathbf{t}]\!]).\mathrm{e}[\![\mathrm{x}]\!]) \diamondsuit [!\mathrm{u}.\lambda\mathrm{x}_0{:}\mathrm{A}[\![\mathrm{u}]\!].\mathrm{e}_0[\![\mathrm{u},\mathrm{x}_0,\mathrm{z}]\!]] = \gamma\mathrm{z}{:}\neg\mathrm{C}.\mathrm{e}[\![\mathrm{x}{:=}\mathrm{c}]\!]\ [:\ \mathrm{C}]$,
if $\Gamma \Vdash \mathbf{t} :: \mathbf{U}$, $\Gamma[\mathrm{x}{:}\neg\mathrm{A}[\![\mathbf{t}]\!]] \vdash \mathrm{e}[\![\mathrm{x}]\!] : \bot$, $\Gamma[\mathrm{u}{:}\mathbf{U}][\mathrm{x}_0{:}\mathrm{A}[\![\mathrm{u}]\!]][\mathrm{z}{:}\neg\mathrm{C}] \vdash \mathrm{e}_0[\![\mathrm{u},\mathrm{x}_0,\mathrm{z}]\!] : \bot$, provided $\mathrm{u} \notin \mathrm{FV}_u(\mathrm{C})$,
where **t** is free for u in $\mathrm{e}_0[\![\mathrm{u},\mathrm{x}_0,\mathrm{z}]\!]$, and $\mathrm{c}[\![\mathrm{z}]\!] \equiv \lambda\mathrm{x}_0{:}\mathrm{A}[\![\mathrm{u}{:=}\mathbf{t}]\!].\mathrm{e}_0[\![\mathrm{u}{:=}\mathbf{t}]\!][\![\mathrm{x}_0,\mathrm{z}]\!]$ is free for x in $\mathrm{e}[\![\mathrm{x}]\!]$,

(122) "∃-*extensionality*":

[η∃] $\Gamma \vdash \bigvee_{\cup}(\mathrm{z}{:}\neg(\exists\mathrm{u}.\mathrm{A}[\![\mathrm{u}]\!])).\mathrm{f} \diamondsuit [!\mathrm{u}.\lambda\mathrm{x}{:}\mathrm{A}[\![\mathrm{u}]\!].\mathrm{z}([\mathrm{u},\mathrm{x}{:}\mathrm{A}[\![\mathrm{u}]\!]])] = \mathrm{f}$, if $\Gamma \vdash \mathrm{f} : \exists\mathrm{u}.\mathrm{A}[\![\mathrm{u}]\!]$,
where $[\mathrm{u},\ \mathrm{x}{:}\mathrm{A}[\![\mathrm{u}]\!]] \equiv \mathbf{J}(\mathrm{x}_1{:}\neg\mathrm{A}[\![\mathrm{u}]\!]).\mathrm{x}_1(\mathrm{x})$, $[\mathrm{u} \notin \mathrm{FV}_u(\exists\mathrm{u}.\mathrm{A}[\![\mathrm{u}]\!])]$.

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[21] Approximating, in traditional terms, these would be applications of "introduction"-rules, immediately followed by applications of *corresponding* "elimination"-rules. The meaning of the "correspondence" should be clear, for the [∨,∃]-*détours*, since we agree, on *this* point, with the traditional "*int-elim*" scheme.

(2) *Normal reductio rules* ($\beta\gamma$-[$\vee$,$\exists$]-*rules*):

[$^h\beta\gamma\vee$] $\Gamma \vdash \gamma$z:$\neg$(A$\vee$B).e[[x:=a]][[y:=b]] = $\mathbf{j}$(x:$\neg$A,y:$\neg$B).e[[x,y]] [: A $\vee$ B], if Γ[x:$\neg$A][y:$\neg$B] $\vdash$ e[[x,y]] : $\perp$,
where
a[[z]] $\equiv \lambda x_0$:A.z($\mathbf{j}_1$[[A,B]](x_0)) $\equiv \lambda x_0$:A.z($\mathbf{j}$(x:$\neg$A,y:$\neg$B).x(x_0)) is free for x in e[[x,y]], and
b[[z]] $\equiv \lambda y_0$:B.z($\mathbf{j}_2$[[A,B]](y_0)) $\equiv \lambda y_0$:B.z($\mathbf{j}$(x:$\neg$A,y:$\neg$B).y(y_0)) is free for y in e[[x,y]],

[$^h\beta\gamma\exists$] $\Gamma \vdash \gamma$z:$\neg$($\exists$u.A[[u]]).e[[x:=a]] = $\mathbf{J}$(x:$\neg$A[[$\mathbf{t}$]]).e[[x]] [: $\exists$u.A[[u]]], if $\Gamma \Vdash \mathbf{t}$ :: $\mathbf{U}$, Γ[x:$\neg$A[[$\mathbf{t}$]]] $\vdash$ e[[x]] : $\perp$,
where a[[z]] $\equiv \lambda x_0$:A[[$\mathbf{t}$]].z([$\mathbf{t}$,x_0:A[[$\mathbf{t}$]]]) $\equiv \lambda x_0$:A[[$\mathbf{t}$]].z($\mathbf{J}$(x:$\neg$A[[$\mathbf{t}$]]).x(x_0)).

(3) *Congruence rules*.

(31) $\vee$-*congruence*.

[$\xi\vee\mathbf{j}$] $\Gamma \vdash \mathbf{j}$(x:$\neg$A,y:$\neg$B).e_1[[x,y]] = $\mathbf{j}$(x:$\neg$A,y:$\neg$B).e_2[[x,y]] [: A $\vee$ B],
if Γ[x:$\neg$A][y:$\neg$B] $\vdash$ e_1[[x,y]] = e_2[[x,y]] : $\perp$,

[$\xi\vee\bigvee_\natural$] $\Gamma \vdash \bigvee_\natural$(z:$\neg$C).f $\diamondsuit$ [λx:A.a, λy:B.b] = $\bigvee_\natural$(z:$\neg$C).g $\diamondsuit$ [λx:A.c, λy:B.d] [: C],
if $\Gamma \vdash$ f = g : A $\vee$ B, Γ[x:A][z:$\neg$C] $\vdash$ a[[x,z]] = c[[x,z]] : $\perp$, Γ[y:B][z:$\neg$C] $\vdash$ b[[y,z]] = d[[y,z]] : $\perp$,

(32) $\exists$-*congruence*.

[$\xi\exists\mathbf{J}$] $\Gamma \vdash \mathbf{J}$(x:$\neg$A[[$\mathbf{t}$]]).e_1[[x]] = $\mathbf{J}$(x:$\neg$A[[$\mathbf{t}$]]).e_2[[x]] [: $\exists$u.A[[u]]],
if $\Gamma \Vdash \mathbf{t}$ :: $\mathbf{U}$, Γ[x:$\neg$A[[$\mathbf{t}$]]] $\vdash$ e_1[[x]] = e_2[[x]] : $\perp$, provided [u:$\mathbf{U}$] is not in Γ,

[$\xi\exists\bigvee_\cup$] $\Gamma \vdash \bigvee_\cup$(z:$\neg$C).f $\diamondsuit$ [!u.λx:A[[u]].e_1] = $\bigvee_\cup$(z:$\neg$C).g $\diamondsuit$ [!u.λx:A[[u]].e_2] [: C],
if $\Gamma \vdash$ f = g : $\exists$u.A $\vee$ B, Γ[u:$\mathbf{U}$][x:A[[u]]][z:$\neg$C] $\vdash$ e_1[[u,x,z]] = e_2[[u,x,z]] : $\perp$, provided u $\notin$ FV_u(C).

Proof. Routine [*exercise*]. The *explicit* derivations should make clear the fact that the diagonalization-free fragment of $\lambda\gamma_\&$**CQ** (a proof-system $\lambda\gamma\pi!$, say) is sufficient. Ignoring the congruence rules, the minimal $\lambda\gamma_\&$**CQ**-derivability conditions are:

(1) *Evaluation rules*:

[$\beta\vee$]: [$\beta\to\lambda$], [$\beta\wedge_1$], [$\beta\wedge_2$], [$\eta\vee$]: [$\beta\to\lambda$], [$\eta\to\lambda$], [$\eta\wedge$], [$\beta\gamma\perp$], [$^h\beta\gamma\to$],
[$\beta\exists$]: [$\beta\to\lambda$], [$\beta\forall$], [$\eta\exists$]: [$\beta\to\lambda$], [$\eta\to\lambda$], [$\eta\forall$], [$\beta\gamma\perp$], [$^h\beta\gamma\to$].

(2) *Reductio rules*:

[$^h\beta\gamma\vee$]: [$\beta\to\lambda$], [$\eta\to\lambda$], [$\beta\gamma\perp$], [$^h\beta\gamma\to$], [$^h\beta\gamma\exists$]: [$\beta\to\lambda$], [$\eta\to\lambda$], [$\beta\gamma\perp$], [$^h\beta\gamma\to$].

(3) *Congruence rules*: by the congruence rules of $\lambda\gamma_\&$**CQ**. $\square$

The negative Boolean selectors: *equational behavior*. In a certain sense, the negative Boolean *selectors* $\bigvee_\natural$, and $\bigvee_\cup$, resp. behave like the negative Boolean *inferential sumptor* (i.e., the γ-abstractor). Indeed, there are β-*structural analogies* [$\gamma \equiv \gamma_\vdash \simeq \bigvee_\natural$], [$\gamma \equiv \gamma_\vdash \simeq \bigvee_\cup$] that can be catalogued systematically.

More generally, from the point of view of *certain* well-behaved *sub-systems* of classical logic, – "positively structured" on the inferential level, so to speak – the *inferential sumptor* $\gamma_\vdash$ should have rather been primitively classified as [being] an *inferential selector* $\bigvee_\vdash$, say (we don't have this in the primitive proof-syntax, here). [E.g., the *special case* of γ which makes sense for the intuitionist – viz. ω (the *ex falso*-family) below, where ω_A(e) $\equiv \gamma$z:$\neg$A.e, [z $\notin$ FV_λ(e)] – is to be viewed as a genuine *selector*, within the Heyting proof-calculus.] As an alternative *epistemic decision*, one could choose to define a [Boolean] $\lambda\gamma$-calculus $\lambda\gamma_{\{*\}}$**CQ**, say, (on [$\perp$,$\to$,$\wedge$,$\vee$,$\forall$,$\exists$]), "symmetrically", with *both* a negative inferential sumptor $\gamma_\vdash$ *and* an additional negative inferential selector $\bigvee_\vdash$, as proof-primitives. Technically, this would increase considerably the number of the "negative proof-isomorphisms" required in order to characterize the new "duality" [as well as the induced interactions with the remaining proof-operators].

From this we retain, in what follows, only the information relevant in the analysis of **HQ**.

Remark ([$\beta\bigvee_\natural\perp$]-*and* [$\beta\bigvee_\cup\perp$]-*rules in* $\lambda\gamma_\&$**CQ**). The following [$\bigvee_\natural$,$\bigvee_\cup$]-analogues of [$\beta\gamma\perp$] are derivable in $\lambda\gamma_\&$**CQ**, using [$\beta\gamma\perp$] (note that the hypotheses yield x, y, z, z_0 $\notin$ FV_λ(h)):

$[\beta\bigvee_\natural\bot]$ $\Gamma \vdash \gamma z{:}\neg C.(\bigvee_\natural(z_0{:}\top).h \diamondsuit [\lambda x{:}A.e_1,\ \lambda y{:}B.e_2]) = \bigvee_\natural(z{:}\neg C).h \diamondsuit [\lambda x{:}A.c_1,\ \lambda y{:}B.c_2]\ [{:}\ C]$,
if $\Gamma \vdash h : A \vee B$, $\Gamma[x{:}A][z{:}\neg C][z_0{:}\top] \vdash e_1\llbracket x,z,z_0\rrbracket : \bot$, $\Gamma[y{:}B][z{:}\neg C][z_0{:}\top] \vdash e_2\llbracket y,z,z_0\rrbracket : \bot$,
where $c_1\llbracket x,z\rrbracket \equiv e_1\llbracket x,z,z_0\rrbracket\llbracket z_0{:=}\Omega\rrbracket$, and $c_2\llbracket x,z\rrbracket \equiv e_2\llbracket x,z,z_0\rrbracket\llbracket z_0{:=}\Omega\rrbracket$,

$[\beta\bigvee_\cup\bot]$ $\Gamma \vdash \gamma z{:}\neg C.(\bigvee_\cup(z_0{:}\top).h \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.e]) = \bigvee_\cup(z{:}\neg C).h \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.c]\ [{:}\ C]$,
if $\Gamma \vdash h : \exists u.A\llbracket u\rrbracket$, $\Gamma[u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket][z{:}\neg C][z_0{:}\top] \vdash e\llbracket u,x,z,z_0\rrbracket : \bot$, provided $u \notin FV_u(C)$,
where $c\llbracket u,x,z\rrbracket \equiv e\llbracket u,x,z,z_0\rrbracket\llbracket z_0{:=}\Omega\rrbracket$.

The following analogues of the $\beta\gamma$-rules are derivable in $\lambda\gamma_{\&}\mathbf{CQ}$.

Theorem (*Positive* $[\beta\bigvee_\natural]$-*and* $[\beta\bigvee_\cup]$-*rules in* $\lambda\gamma_{\&}\mathbf{CQ}$).

(1) *Positive* $[\beta\bigvee_\natural]$-*rules*:

$[\beta\bigvee_\natural\to]$ $\Gamma \vdash (\bigvee_\natural(z{:}\neg(F\to G)).h \diamondsuit [\lambda x{:}A.e_1,\ \lambda y{:}B.e_2])(f) = \bigvee_\natural(z_1{:}\neg G).h \diamondsuit [\lambda x{:}A.c_1,\ \lambda y{:}B.c_2]\ [{:}\ G]$,
if $\Gamma \vdash h : A \vee B$, $\Gamma \vdash f : F$,
and $\Gamma[x{:}A][z{:}\neg(F\to G)] \vdash e_1\llbracket x,z\rrbracket : \bot$, $\Gamma[y{:}B][z{:}\neg(F\to G)] \vdash e_2\llbracket y,z\rrbracket : \bot$,
where $[x, y, z, z_1 \notin FV_\lambda(f,h)]$, $[z_1 \notin FV_\lambda(e_1\llbracket x,z\rrbracket,e_2\llbracket y,z\rrbracket)]$,
$c_1\llbracket x,z_1\rrbracket \equiv e_1\llbracket x\rrbracket\llbracket z{:=}\lambda z{:}(F\to G).z_1(z(f))\rrbracket$, and $c_2\llbracket y,z_1\rrbracket \equiv e_2\llbracket y\rrbracket\llbracket z{:=}\lambda z{:}(F\to G).z_1(z(f))\rrbracket$,

$[\beta\bigvee_\natural\wedge_1]$ $\Gamma \vdash \mathbf{p}_1(\bigvee_\natural(z{:}\neg(F\wedge G)).h \diamondsuit [\lambda x{:}A.e_1,\ \lambda y{:}B.e_2]) = \bigvee_\natural(z_1{:}\neg F).h \diamondsuit [\lambda x{:}A.c_1,\ \lambda y{:}B.c_2]\ [{:}\ F]$,
if $\Gamma \vdash h : A \vee B$, $\Gamma[x{:}A][z{:}\neg(F\wedge G)] \vdash e_1\llbracket x,z\rrbracket : \bot$, $\Gamma[y{:}B][z{:}\neg(F\wedge G)] \vdash e_2\llbracket y,z\rrbracket : \bot$,
where $[x, y, z, z_1 \notin FV_\lambda(h)]$, $[z_1 \notin FV_\lambda(e_1\llbracket x,z\rrbracket,e_2\llbracket y,z\rrbracket)]$,
$c_1\llbracket x,z_1\rrbracket \equiv e_1\llbracket x\rrbracket\llbracket z{:=}\lambda z{:}(F\wedge G).z_1(\mathbf{p}_1(z))\rrbracket$, and $c_2\llbracket y,z_1\rrbracket \equiv e_2\llbracket y\rrbracket\llbracket z{:=}\lambda z{:}(F\wedge G).z_1(\mathbf{p}_1(z))\rrbracket$,

$[\beta\bigvee_\natural\wedge_2]$ $\Gamma \vdash \mathbf{p}_2(\bigvee_\natural(z{:}\neg(F\wedge G)).h \diamondsuit [\lambda x{:}A.e_1,\ \lambda y{:}B.e_2]) = \bigvee_\natural(z_2{:}\neg G).h \diamondsuit [\lambda x{:}A.c_1,\ \lambda y{:}B.c_2]\ [{:}\ G]$,
if $\Gamma \vdash h : A \vee B$, $\Gamma[x{:}A][z{:}\neg(F\wedge G)] \vdash e_1\llbracket x,z\rrbracket : \bot$, $\Gamma[y{:}B][z{:}\neg(F\wedge G)] \vdash e_2\llbracket y,z\rrbracket : \bot$,
where $[x, y, z, z_2 \notin FV_\lambda(h)]$, $[z_2 \notin FV_\lambda(e_1\llbracket x,z\rrbracket,e_2\llbracket y,z\rrbracket)]$,
$c_1\llbracket x,z_2\rrbracket \equiv e_1\llbracket x\rrbracket\llbracket z{:=}\lambda z{:}(F\wedge G).z_2(\mathbf{p}_2(z))\rrbracket$, and $c_2\llbracket y,z_2\rrbracket \equiv e_2\llbracket y\rrbracket\llbracket z{:=}\lambda z{:}(F\wedge G).z_2(\mathbf{p}_2(z))\rrbracket$,

$[\beta\bigvee_\natural\forall]$ $\Gamma \vdash (\bigvee_\natural(z{:}\neg(\forall v.F\llbracket v\rrbracket)).h \diamondsuit [\lambda x{:}A.e_1,\ \lambda y{:}B.e_2])[\mathbf{t}] = \bigvee_\natural(z_1{:}\neg F\llbracket\mathbf{t}\rrbracket).h \diamondsuit [\lambda x{:}A.c_1,\ \lambda y{:}B.c_2]\ [{:}\ F\llbracket\mathbf{t}\rrbracket]$,
if $\Gamma \vdash h : A \vee B$, $\Gamma \Vdash \mathbf{t} :: \mathbf{U}$,
and $\Gamma[x{:}A][z{:}\neg(\forall v.F\llbracket v\rrbracket)] \vdash e_1\llbracket x,z\rrbracket : \bot$, $\Gamma[y{:}B][z{:}\neg(\forall v.F\llbracket v\rrbracket)] \vdash e_2\llbracket y,z\rrbracket : \bot$,
where $[x, y, z, z_1 \notin FV_\lambda(h)]$, $[z_1 \notin FV_\lambda(e_1\llbracket x,z\rrbracket,e_2\llbracket y,z\rrbracket)]$, $[v \notin FV_u(A,B,h,e_1,e_2)]$,
$F\llbracket\mathbf{t}\rrbracket \equiv F\llbracket v{:=}\mathbf{t}\rrbracket$, with $\mathbf{t}$ free for v in $F\llbracket v\rrbracket$,
$c_1\llbracket x,z_1\rrbracket \equiv e_1\llbracket x\rrbracket\llbracket z{:=}\lambda z{:}(\forall v.F\llbracket v\rrbracket).z_1(z[\mathbf{t}])\rrbracket$, and $c_2\llbracket x,z_1\rrbracket \equiv e_2\llbracket y\rrbracket\llbracket z{:=}\lambda z{:}(\forall v.F\llbracket v\rrbracket).z_1(z[\mathbf{t}])\rrbracket$.

(2) *Positive* $[\beta\bigvee_\cup]$-*rules*:

$[\beta\bigvee_\cup\to]$ $\Gamma \vdash (\bigvee_\cup(z{:}\neg(F\to G)).h \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.e])(f) = \bigvee_\cup(z_1{:}\neg G).h \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.c]\ [{:}\ G]$,
if $\Gamma \vdash f : F$, $\Gamma \vdash h : \exists u.A\llbracket u\rrbracket$, $\Gamma[u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket][z{:}\neg(F\to G)] \vdash e\llbracket u,x,z\rrbracket : \bot$,
where $[x, z, z_0, z_1 \notin FV_\lambda(f,h)]$, $[z_1 \notin FV_\lambda(e\llbracket u,x,z\rrbracket)]$,
$c\llbracket u,x,z_1\rrbracket \equiv e\llbracket u,x\rrbracket\llbracket z{:=}\lambda z{:}(F\to G).z_1(z(f))\rrbracket$, $[u \notin FV_u(F,G)]$,

$[\beta\bigvee_\cup\wedge_1]$ $\Gamma \vdash \mathbf{p}_1(\bigvee_\cup(z{:}\neg(F\wedge G)).h \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.e]) = \bigvee_\cup(z_1{:}\neg F).h \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.c]\ [{:}\ F]$,
if $\Gamma \vdash h : \exists u.A\llbracket u\rrbracket$, $\Gamma[u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket][z{:}\neg(F\wedge G)] \vdash e\llbracket u,x,z\rrbracket : \bot$,
where $[x, z, z_1 \notin FV_\lambda(h)]$, $[z_1 \notin FV_\lambda(e\llbracket u,x,z\rrbracket)]$,
$c\llbracket u,x,z_1\rrbracket \equiv e\llbracket u,x\rrbracket\llbracket z{:=}\lambda z{:}(F\wedge G).z_1(\mathbf{p}_1(z))\rrbracket$, $[u \notin FV_u(F,G)]$,

$[\beta\bigvee_\cup\wedge_2]$ $\Gamma \vdash \mathbf{p}_2(\bigvee_\cup(z{:}\neg(F\wedge G)).h \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.e]) = \bigvee_\cup(z_2{:}\neg G).h \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.c]\ [{:}\ G]$,
if $\Gamma \vdash h : \exists u.A\llbracket u\rrbracket$, $\Gamma[u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket][z{:}\neg(F\wedge G)] \vdash e\llbracket u,x,z\rrbracket : \bot$,
where $[x, z, z_2 \notin FV_\lambda(h)]$, $[z_2 \notin FV_\lambda(e\llbracket u,x,z\rrbracket)]$,
$c\llbracket u,x,z_2\rrbracket \equiv e\llbracket u,x\rrbracket\llbracket z{:=}\lambda z{:}(F\wedge G).z_2(\mathbf{p}_2(z))\rrbracket$, $[u \notin FV_u(F,G)]$,

$[\beta\bigvee_\cup\forall]$ $\Gamma \vdash (\bigvee_\cup(z{:}\neg(\forall v.F\llbracket v\rrbracket)).h \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.e])[\mathbf{t}] = \bigvee_\cup(z_1{:}\neg F\llbracket\mathbf{t}\rrbracket).h \diamondsuit [!u.\lambda x{:}A\llbracket u\rrbracket.c]\ [{:}\ F\llbracket\mathbf{t}\rrbracket]$,
if $\Gamma \Vdash \mathbf{t} :: \mathbf{U}$, $\Gamma \vdash h : \exists u.A\llbracket u\rrbracket$, $\Gamma[u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket][z{:}\neg(\forall v.F\llbracket v\rrbracket)] \vdash e\llbracket u,x,z\rrbracket : \bot$,
where $[x, z, z_1 \notin FV_\lambda(h)]$, $[z_1 \notin FV_\lambda(e\llbracket u,x,z\rrbracket)]$,

$F[\![\mathbf{t}]\!] \equiv F[\![v{:=}\mathbf{t}]\!]$, with $\mathbf{t}$ free for v in $F[\![v]\!]$.
$c[\![u,x,z_1]\!] \equiv e[\![u,x]\!][\![z{:=}\lambda z{:}(\forall v.F[\![v]\!]).z_1(z[\mathbf{t}])]\!]$, $[u \notin FV_u(\forall v.F[\![v]\!])]$.

Proof. Straightforward [*exercise*] (use the $\beta\gamma$-rules $[\beta\gamma \to]$, $[\beta\gamma\wedge_1]$, $[\beta\gamma\wedge_2]$, and $[\beta\gamma\forall]$, resp.) □

Remark (*Extensional* $[\beta\bigvee_\natural]$- *and* $[\beta\bigvee_\cup]$-*lifting*). It is easy to see that "extensionally lifted" analogues $[^h\beta\bigvee_\natural\to]$, $[^h\beta\bigvee_\natural\wedge]$, $[^h\beta\bigvee_\natural\forall]$ and $[^h\beta\bigvee_\cup\to]$, $[^h\beta\bigvee_\cup\wedge]$, $[^h\beta\bigvee_\cup\forall]$, resp. of the positive $[\beta\bigvee_\natural]$- and $[\beta\bigvee_\cup]$-rules: $[\beta\bigvee_\natural\to]$, $[\beta\bigvee_\natural\wedge_1]$, $[\beta\bigvee_\natural\wedge_2]$, $[\beta\bigvee_\natural\forall]$ and $[\beta\bigvee_\cup\to]$, $[\beta\bigvee_\cup\wedge_1]$, $[\beta\bigvee_\cup\wedge_2]$, $[\beta\bigvee_\cup\forall]$ resp. are also derivable in $\lambda\gamma_\&\mathbf{CQ}$. The statement and the derivation of the "lifted" analogues of the $[\beta\bigvee_\natural]$- and $[\beta\bigvee_\cup]$-rules are left as *exercises*. [These rules are perhaps more dificult to state than to prove!]

Remark (*The* β-$[\bigvee_\natural,\bigvee_\cup]$-*rules*: $\lambda\gamma_\&\mathbf{CQ}$-*derivability conditions*). By inspecting the corresponding *explicit* derivations, one finds that the $[\beta\bigvee_\natural]$- and $[\beta\bigvee_\cup]$-rules above are derivable in a diagonalization-free fragment of $\lambda\gamma_\&\mathbf{CQ}$ without $[\wedge,\forall]$-"extensionality". However, the derivation of the "lifted" variants $[^h\beta\bigvee_\natural\wedge]$, $[^h\beta\bigvee_\cup\wedge]$ and $[^h\beta\bigvee_\natural\forall]$, $[^h\beta\bigvee_\cup\forall]$ resp. require $[\eta\wedge]$ and $[\eta\forall]$, resp.

As shown below, the β-$[\bigvee_\natural,\bigvee_\cup]$-rules specialize to "positive Heyting $[\vee,\exists]$-commuting rules".

Remark (*Special cases of the* $[\bigvee_\natural,\bigvee_\cup]$-*selectors*). The following equivalences are also derivable, as limit cases, in $\lambda\gamma_\&\mathbf{CQ}$, using $[\beta\gamma\bot]$ and $[\beta\to\lambda]$, [*exercise*]:

$[\bigvee_\natural\sqcup_m{:}\bot]$ $\Gamma \vdash \bigvee_\natural(z{:}\top).h \diamondsuit [\lambda x{:}A.e_1,\ \lambda y{:}B.e_2] = \sqcup(h,[x{:}A].e_1{:}\bot,[y{:}B].e_2{:}\bot)\ [{:}\ \bot]$,
if $\Gamma \vdash h : A \vee B$, $\Gamma[x{:}A] \vdash e_1[\![x]\!] : \bot$, $\Gamma[y{:}B] \vdash e_2[\![y]\!] : \bot$,
provided $z \notin FV_\lambda(e_1[\![x]\!],e_2[\![y]\!])$, $[x, y, z \notin FV_\lambda(h)]$,

$[\bigvee_\cup II_m{:}\bot]$ $\Gamma \vdash \bigvee_\cup(z{:}\top).h \diamondsuit [!u.\lambda x{:}A[\![u]\!].e] = II(h,[u{:}\mathbf{U}][x{:}A[\![u]\!]].e{:}\bot)\ [{:}\ \bot]$,
if $\Gamma \vdash h : \exists u.A[\![u]\!]$, $\Gamma[u{:}\mathbf{U}][x{:}A[\![u]\!]] \vdash e[\![u,x]\!] : \bot$,
provided $z \notin FV_\lambda(e[\![u,x]\!])$, $[x, z \notin FV_\lambda(h)]$, $[u \notin FV_u(h)]$.

Remark (*Local* $[\mathbf{j},\bigvee_\natural]$-*extensionality in* $\lambda\gamma_\&\mathbf{CQ}$). Although the $\exists$-pair of proof-operators $[\mathbf{J},\bigvee_\cup]$ generalizes the $\vee$-pair $[\mathbf{j},\bigvee_\natural]$, they differ, in $\lambda\gamma_\&\mathbf{CQ}$, on extensionality properties. E.g., where x, y, z, x_0, $y_0 \notin FV_\lambda(f)$, one has, using $[^h\beta\gamma\vee]$, $[\beta\to\lambda]$, $[\beta\bigvee_\natural\bot]$ and $[\eta\vee]$,

$[\eta\vee\mathbf{j}]$ $\Gamma \vdash \mathbf{j}(x{:}\neg A,y{:}\neg B).\bigvee_\natural(z{:}\top).f \diamondsuit [\lambda x_0{:}A.x(x_0),\lambda y_0{:}B.y(y_0)] = f$, if $\Gamma \vdash f : A \vee B$,

(which can be further simplified, by $[\bigvee_\natural\sqcup_m{:}\bot]$) [*exercise*]. There is no $\exists$-analogue of this (for $[\mathbf{J},\bigvee_\cup]$), however.

$\bigvee_\natural$- *and* $\bigvee_\cup$-*diagonal situations in* $\lambda\gamma_\&\mathbf{CQ}$. Finally, we examine the mixed $\bigvee_\natural$- and $\bigvee_\cup$-analogues of the γ-diagonalization rule $[\oint\gamma]$.

The following "simple diagonalization" rules are derivable in $\lambda\gamma_\&\mathbf{CQ}$.

Lemma (*Simple* $\bigvee_\natural$- *and* $\bigvee_\cup$-*diagonalization in* $\lambda\gamma_\&\mathbf{CQ}$).

(1) $[\oint\bigvee_\natural\gamma]$-*rules*: For x, y, z, $z_0 \notin FV_\lambda(h)$,

$[\oint\bigvee_\natural\gamma\bot]$
$\Gamma \vdash \bigvee_\natural(z{:}\neg C).h \diamondsuit [\lambda x{:}A.f(z(\gamma z_0{:}\neg C.e_1)),\ \lambda y{:}B.e_2] = \bigvee_\natural(z{:}\neg C).h \diamondsuit [\lambda x{:}A.f(e_1[\![x,z,z]\!]),\ \lambda y{:}B.e_2]\ [{:}\ C]$,
if $\Gamma \vdash h : A \vee B$, $\Gamma[x{:}A][z{:}\neg C] \vdash f[\![x,z]\!] : \top$,
and $\Gamma[x{:}A][z{:}\neg C][z_0{:}\neg C] \vdash e_1[\![x,z,z_0]\!] : \bot$, $\Gamma[y{:}B][z{:}\neg C] \vdash e_2[\![y,z]\!] : \bot$,
where $e_1[\![x,z,z]\!] \equiv e_1[\![x,z,z_0]\!][\![z_0{:=}z]\!]$, $[z_0 \notin FV_\lambda(f[\![x,z]\!],e_2[\![y,z]\!])]$,

$[\oint\bigvee_\natural\bot\gamma]$
$\Gamma \vdash \bigvee_\natural(z{:}\neg C).h \diamondsuit [\lambda x{:}A.e_1,\ \lambda y{:}B.g(z(\gamma z_0{:}\neg C.e_2))] = \bigvee_\natural(z{:}\neg C).h \diamondsuit [\lambda x{:}A.e_1,\ \lambda y{:}B.g(e_2[\![y,z,z]\!])]\ [{:}\ C]$,
if $\Gamma \vdash h : A \vee B$, $\Gamma[y{:}B][z{:}\neg C] \vdash g[\![y,z]\!] : \top$,
and $\Gamma[x{:}A][z{:}\neg C] \vdash e_1[\![x,z]\!] : \bot$, $\Gamma[y{:}B][z{:}\neg C][z_0{:}\neg C] \vdash e_2[\![y,z,z_0]\!] : \bot$,
where $e_2[\![y,z,z]\!] \equiv e_2[\![y,z,z_0]\!][\![z_0{:=}z]\!]$, $[z_0 \notin FV_\lambda(e_1[\![x,z]\!],g[\![y,z]\!])]$,

$[\oint\bigvee_\natural\gamma\gamma]$
$\Gamma \vdash \bigvee_\natural(\mathrm{z}{:}\neg\mathrm{C}).\mathrm{h} \Diamond [\lambda\mathrm{x{:}A.f}(\mathrm{z}(\gamma \mathrm{z}_0{:}\neg\mathrm{C.e}_1)),\lambda\mathrm{y{:}B.g}(\mathrm{z}(\gamma\mathrm{z}_0{:}\neg\mathrm{C.e}_2))] = \bigvee_\natural(\mathrm{z}{:}\neg\mathrm{C}).\mathrm{h} \Diamond [\lambda\mathrm{x{:}A.fc}_1,\lambda\mathrm{y{:}B.gc}_2]$ [: C],
if $\Gamma \vdash \mathrm{h} : \mathrm{A} \vee \mathrm{B}$, $\Gamma[\mathrm{x{:}A}][\mathrm{z}{:}\neg\mathrm{C}] \vdash \mathrm{f}\llbracket\mathrm{x,z}\rrbracket : \top$, $\Gamma[\mathrm{y{:}B}][\mathrm{z}{:}\neg\mathrm{C}] \vdash \mathrm{g}\llbracket\mathrm{y,z}\rrbracket : \top$,
$\Gamma[\mathrm{x{:}A}][\mathrm{z}{:}\neg\mathrm{C}][\mathrm{z}_0{:}\neg\mathrm{C}] \vdash \mathrm{e}_1\llbracket\mathrm{x,z,z}_0\rrbracket : \bot$, $\Gamma[\mathrm{y{:}B}][\mathrm{z}{:}\neg\mathrm{C}][\mathrm{z}_0{:}\neg\mathrm{C}] \vdash \mathrm{e}_2\llbracket\mathrm{y,z,z}_0\rrbracket : \bot$,
where $\mathrm{c}_1 \equiv \mathrm{e}_1\llbracket\mathrm{x,z,z}\rrbracket \equiv \mathrm{e}_1\llbracket\mathrm{x,z,z}_0\rrbracket\llbracket\mathrm{z}_0{:=}\mathrm{z}\rrbracket$, $\mathrm{c}_2 \equiv \mathrm{e}_2\llbracket\mathrm{y,z,z}\rrbracket \equiv \mathrm{e}_2\llbracket\mathrm{y,z,z}_0\rrbracket\llbracket\mathrm{z}_0{:=}\mathrm{z}\rrbracket$, $[\mathrm{z}_0 \notin \mathrm{FV}_\lambda(\mathrm{f}\llbracket\mathrm{x,z}\rrbracket,\mathrm{g}\llbracket\mathrm{y,z}\rrbracket)]$,

(2) $[\oint\bigvee_\cup\gamma]$-*rule*: For x, y, z, $\mathrm{z}_0 \notin \mathrm{FV}_\lambda(\mathrm{h})$, and $\mathrm{z}_0 \notin \mathrm{FV}_\lambda(\mathrm{f}\llbracket\mathrm{u,x,z}\rrbracket)$,

$[\oint\bigvee_\cup\gamma]$
$\Gamma \vdash \bigvee_\cup(\mathrm{z}{:}\neg\mathrm{C}).\mathrm{h} \Diamond [!\mathrm{u}.\lambda\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket.\mathrm{f}(\mathrm{z}(\gamma\mathrm{z}_0{:}\neg\mathrm{C.e}\llbracket\mathrm{u,x,z,z}_0\rrbracket)))] = \bigvee_\cup(\mathrm{z}{:}\neg\mathrm{C}).\mathrm{h} \Diamond [!\mathrm{u}.\lambda\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket.\mathrm{f}(\mathrm{e}\llbracket\mathrm{u,x,z,z}\rrbracket)]$ [: C],
if $\Gamma \vdash \mathrm{h} : \exists\mathrm{u.A}\llbracket\mathrm{u}\rrbracket$, $\Gamma[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket][\mathrm{z}{:}\neg\mathrm{C}] \vdash \mathrm{f}\llbracket\mathrm{u,x,z}\rrbracket : \top$,
and $\Gamma[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket][\mathrm{z}{:}\neg\mathrm{C}][\mathrm{z}_0{:}\neg\mathrm{C}] \vdash \mathrm{e}\llbracket\mathrm{u,x,z,z}_0\rrbracket : \bot$, provided $\mathrm{u} \notin \mathrm{FV}_u(\mathrm{C})$,
where $\mathrm{e}\llbracket\mathrm{u,x,z,z}\rrbracket \equiv \mathrm{e}\llbracket\mathrm{u,x,z,z}_0\rrbracket\llbracket\mathrm{z}_0{:=}\mathrm{z}\rrbracket$,

Proof. Straightforward [*exercise*] (use $[\oint\gamma]$). □

The following consequences of $[\oint\bigvee_\natural\gamma\gamma]$ and $[\oint\bigvee_\cup\gamma]$ (*modulo* $[\beta\bigvee_\natural\bot]$ and $[\beta\bigvee_\cup\bot]$) are the general Boolean variants of the Heyting $[\vee\omega]$- and $[\exists\omega]$-"commuting conversions".

Corollary (*$[\beta\vee\gamma]$- and $[\beta\exists\gamma]$-conversions*).

(1) $[\beta\bigvee_\natural\bot\text{-}\oint\bigvee_\natural\gamma\gamma]$-*conversion*: For $\mathrm{z}_1 \notin \mathrm{FV}_\lambda(\mathrm{e}_1\llbracket\mathrm{x,z,z}_0\rrbracket)$, and $\mathrm{z}_1 \notin \mathrm{FV}_\lambda(\mathrm{e}_2\llbracket\mathrm{y,z,z}_0\rrbracket)$,

$[\beta\vee\gamma]$ $\Gamma \vdash \gamma\mathrm{z}{:}\neg\mathrm{C}.\bigvee_\natural(\mathrm{z}_0{:}\top).\mathrm{h} \Diamond [\lambda\mathrm{x{:}A.e}_1, \lambda\mathrm{y{:}B.e}_2] = \bigvee_\natural(\mathrm{z}{:}\neg\mathrm{C}).\mathrm{h} \Diamond [\lambda\mathrm{x{:}A.c}_1\llbracket\mathrm{x,z}\rrbracket, \lambda\mathrm{y{:}B.c}_2\llbracket\mathrm{y,z}\rrbracket] : \mathrm{C}$,
if $\Gamma \vdash \mathrm{h} : \mathrm{A} \vee \mathrm{B}$, $\Gamma[\mathrm{x{:}A}][\mathrm{z}{:}\neg\mathrm{C}][\mathrm{z}_0{:}\top] \vdash \mathrm{e}_1\llbracket\mathrm{x,z,z}_0\rrbracket : \bot$, $\Gamma[\mathrm{y{:}B}][\mathrm{z}{:}\neg\mathrm{C}][\mathrm{z}_0{:}\top] \vdash \mathrm{e}_2\llbracket\mathrm{y,z,z}_0\rrbracket : \bot$,
where $\mathrm{c}_1\llbracket\mathrm{x,z}\rrbracket \equiv \mathrm{z}(\gamma\mathrm{z}_1{:}\neg\mathrm{C.e}_1\llbracket\mathrm{x,z,z}_0\rrbracket\llbracket\mathrm{z}_0{:=}\Omega\rrbracket)$, $\mathrm{c}_2\llbracket\mathrm{y,z}\rrbracket \equiv \mathrm{z}(\gamma\mathrm{z}_1{:}\neg\mathrm{C.e}_2\llbracket\mathrm{x,z,z}_0\rrbracket\llbracket\mathrm{z}_0{:=}\Omega\rrbracket)$,

(2) $[\beta\bigvee_\cup\bot\text{-}\oint\bigvee_\cup\gamma]$-*conversion*: For $\mathrm{z}_1 \notin \mathrm{FV}_\lambda(\mathrm{e}\llbracket\mathrm{u,x,z,z}_0\rrbracket)$,

$[\beta\exists\gamma]$ $\Gamma \vdash \gamma\mathrm{z}{:}\neg\mathrm{C}.\bigvee_\natural(\mathrm{z}_0{:}\top).\mathrm{h} \Diamond [!\mathrm{u}.\lambda\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket.\mathrm{e}] = \bigvee_\natural(\mathrm{z}{:}\neg\mathrm{C}).\mathrm{h} \Diamond [!\mathrm{u}.\lambda\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket.\mathrm{c}\llbracket\mathrm{u,x,z}\rrbracket]$ [: C],
if $\Gamma \vdash \mathrm{h} : \exists\mathrm{u.A}\llbracket\mathrm{u}\rrbracket$, $\Gamma[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket][\mathrm{z}{:}\neg\mathrm{C}][\mathrm{z}_0{:}\top] \vdash \mathrm{e}\llbracket\mathrm{u,x,z,z}_0\rrbracket : \bot$,
where $\mathrm{c}\llbracket\mathrm{u,x,z}\rrbracket \equiv \mathrm{z}(\gamma\mathrm{z}_1{:}\neg\mathrm{C.e}\llbracket\mathrm{u,x,z,z}_0\rrbracket\llbracket\mathrm{z}_0{:=}\Omega\rrbracket)$.

Proof. For $[\beta\vee\gamma]$: Assume the hypotheses [so that $\mathrm{z}_1 \notin \mathrm{FV}_\lambda(\mathrm{e}_1,\mathrm{e}_2)$]. Then, with

$$\mathrm{c} \equiv \bigvee_\natural(\mathrm{z}{:}\neg\mathrm{C}).\mathrm{h} \Diamond [\lambda\mathrm{x{:}A.e}_1\llbracket\mathrm{x,z}\rrbracket\llbracket\mathrm{z}_0{:=}\Omega\rrbracket, \lambda\mathrm{y{:}B.e}_2\llbracket\mathrm{y,z}\rrbracket\llbracket\mathrm{z}_0{:=}\Omega\rrbracket],$$

one has $\Gamma \vdash (\mathrm{LHS}) = \mathrm{c} : \mathrm{C}$, by $[\beta\bigvee_\natural\bot]$, and $\Gamma \vdash (\mathrm{RHS}) = \mathrm{c} : \mathrm{C}$, by $[\oint\bigvee_\natural\gamma\gamma]$ and $[\beta \to \lambda]$. For $[\beta\exists\gamma]$: analogously, using $[\beta\bigvee_\cup\bot]$, and $[\oint\bigvee_\cup\gamma]$, $[\beta \to \lambda]$, resp. (So, the derivations of $[\beta\vee\gamma]$ and $[\beta\exists\gamma]$ in $\lambda\gamma_\&\mathbf{CQ}$ require $[\oint\gamma]$, too.) □

Finally, we have a number of $[\oint\bigvee_\natural]$- and $[\oint\bigvee_\cup]$-*rules*, describing *cross* $[\bigvee_\natural,\bigvee_\cup]$-*diagonal situations* or "$[\bigvee_\natural,\bigvee_\cup]$-*commuting*" cases *in* $\lambda\gamma_\&\mathbf{CQ}$, when a $\bigvee_{(\natural,\cup)}$-selector occurs within the scope of a $\bigvee_{(\natural,\cup)}$-selector, as its "main (neutral) branch". Schematically:

$$\bigvee_{(\natural,\cup)}(\mathrm{z}{:}\neg\mathrm{E}).(\bigvee_{(\natural,\cup)}(\mathrm{z}_0{:}\neg\mathrm{C}).\mathrm{h} \Diamond \varphi\ll\ldots\mathrm{f}\ldots\gg) \Diamond \psi\ll\ldots\mathrm{g}\ldots\gg.$$

An obvious systematic search discloses 12 (= 8 $[\oint\bigvee_\natural]$ + 4 $[\oint\bigvee_\cup]$) distinct cases in $\lambda\gamma_\&\mathbf{CQ}$. Of these, only 4 (= 2 $[\oint\bigvee_\natural]$ + 2 $[\oint\bigvee_\cup]$) can be seen to admit of proper "positive" instances, specializing to relevant "commuting situations" in the Heyting proof-calculus for **HQ**. Ignoring the remaining cases, we obtain, finally, the following general $\bigvee_\natural$- resp. $\bigvee_\cup$-"*commuting rules*".

Theorem (*Cross* $\bigvee_\natural$*-diagonalization in* $\lambda\gamma_\&\mathbf{CQ}$). For z, $\mathrm{z}_0 \notin \mathrm{FV}_\lambda(\mathrm{h,a}\llbracket\mathrm{x}\rrbracket,\mathrm{b}\llbracket\mathrm{y}\rrbracket,\mathrm{f}\llbracket\mathrm{x}\rrbracket,\mathrm{g}\llbracket\mathrm{y}\rrbracket)$,

$[\oint\bigvee_\natural\bigvee_\natural{:}\vee\vee]$
$\Gamma \vdash \bigvee_\natural(\mathrm{z}{:}\neg\mathrm{E}).\mathrm{c} \Diamond [\lambda\mathrm{x}_0{:}\mathrm{F.e}_1,\lambda\mathrm{y}_0{:}\mathrm{G.e}_2] = \bigvee_\natural(\mathrm{z}{:}\neg\mathrm{E}).\mathrm{h} \Diamond [\lambda\mathrm{x{:}A.f}\llbracket\mathrm{x}\rrbracket(\mathrm{z}(\mathrm{c}_1\llbracket\mathrm{x}\rrbracket)),\lambda\mathrm{y{:}B.g}\llbracket\mathrm{y}\rrbracket(\mathrm{z}(\mathrm{c}_2\llbracket\mathrm{y}\rrbracket))]$ [: E],
if $\Gamma \vdash \mathrm{h} : \mathrm{A} \vee \mathrm{B}$, $\Gamma[\mathrm{x{:}A}] \vdash \mathrm{a}\llbracket\mathrm{x}\rrbracket : \mathrm{F} \vee \mathrm{G}$, $\Gamma[\mathrm{y{:}B}] \vdash \mathrm{b}\llbracket\mathrm{y}\rrbracket : \mathrm{F} \vee \mathrm{G}$, $\Gamma[\mathrm{x{:}A}] \vdash \mathrm{f}\llbracket\mathrm{x}\rrbracket : \top$, $\Gamma[\mathrm{y{:}B}] \vdash \mathrm{g}\llbracket\mathrm{y}\rrbracket : \top$,
and $\Gamma[\mathrm{x}_0{:}\mathrm{F}][\mathrm{z}{:}\neg\mathrm{E}] \vdash \mathrm{e}_1\llbracket\mathrm{x}_0,\mathrm{z}\rrbracket : \bot$, $\Gamma[\mathrm{y}_0{:}\mathrm{G}][\mathrm{z}{:}\neg\mathrm{E}] \vdash \mathrm{e}_2\llbracket\mathrm{y}_0,\mathrm{z}\rrbracket : \bot$,
where
$\mathrm{c} \equiv \bigvee_\natural(\mathrm{z}_0{:}\neg\mathrm{C}).\mathrm{h} \Diamond [\lambda\mathrm{x{:}A.f}\llbracket\mathrm{x}\rrbracket(\mathrm{z}_0(\mathrm{a}\llbracket\mathrm{x}\rrbracket)), \lambda\mathrm{y{:}B.g}\llbracket\mathrm{y}\rrbracket(\mathrm{z}_0(\mathrm{b}\llbracket\mathrm{y}\rrbracket))]$, for $\mathrm{C} \equiv \mathrm{F} \vee \mathrm{G}$, and

$c_1\llbracket x\rrbracket \equiv \bigvee_{\natural}(z{:}\neg E).a\llbracket x\rrbracket \Diamond [\lambda x_0{:}F.e_1\llbracket x_0,z\rrbracket,\ \lambda y_0{:}G.e_2\llbracket y_0,z\rrbracket]$,
$c_2\llbracket y\rrbracket \equiv \bigvee_{\natural}(z{:}\neg E).b\llbracket y\rrbracket \Diamond [\lambda x_0{:}F.e_1\llbracket x_0,z\rrbracket,\ \lambda y_0{:}G.e_2\llbracket y_0,z\rrbracket]$,

$[\oint\bigvee_{\cup}\bigvee_{\natural}{:}\exists\exists]$
$\Gamma \vdash \bigvee_{\cup}(z{:}\neg E).c \Diamond [!u.\lambda x_0{:}F.e] = \bigvee_{\natural}(z{:}\neg E).h \Diamond [\lambda x{:}A.f(z(e_1\llbracket x\rrbracket)),\ \lambda y{:}B.g(z(e_2\llbracket y\rrbracket))]\ [:\ E]$,
if $\Gamma \vdash h : A \vee B$, $\Gamma[x{:}A] \vdash a\llbracket x\rrbracket : \exists u.F\llbracket u\rrbracket$, $\Gamma[y{:}B] \vdash b\llbracket y\rrbracket : \exists u.F\llbracket u\rrbracket$, $\Gamma[x{:}A] \vdash f\llbracket x\rrbracket : \top$, $\Gamma[y{:}B] \vdash g\llbracket y\rrbracket : \top$,
and $\Gamma[u{:}\mathbf{U}][x_0{:}F\llbracket u\rrbracket][z{:}\neg E] \vdash e\llbracket u,x_0,z\rrbracket : \bot$, provided $u \notin FV_u(E)$,

where
$c \equiv \bigvee_{\natural}(z_0{:}\neg C).h \Diamond [\lambda x{:}A.f(z_0(a\llbracket x\rrbracket)),\ \lambda y{:}B.g(z_0(b\llbracket y\rrbracket))]$, for $C \equiv \exists u.F\llbracket u\rrbracket$, and
$e_1\llbracket x\rrbracket \equiv \bigvee_{\cup}(z{:}\neg E).a\llbracket x\rrbracket \Diamond [!u.\lambda x_0{:}F\llbracket u\rrbracket.e\llbracket u,x_0,z\rrbracket]$, $e_2\llbracket y\rrbracket \equiv \bigvee_{\cup}(z{:}\neg E).b\llbracket y\rrbracket \Diamond [!u.\lambda x_0{:}F\llbracket u\rrbracket.e\llbracket u,x_0,z\rrbracket]$.

Proof. Routine. We record the minimal derivability conditions in $\lambda\gamma_{\&}\mathbf{CQ}$, leaving the details to the reader.

For $[\oint\bigvee_{\natural}\bigvee_{\natural}{:}\vee\vee]$: where

$$e' \equiv \gamma z{:}\neg E.h(<\lambda x{:}A.f\llbracket x\rrbracket(a\llbracket x\rrbracket(d\llbracket y\rrbracket)){:}\neg A,\ \lambda y{:}B.g\llbracket y\rrbracket(b\llbracket y\rrbracket(d\llbracket z\rrbracket)){:}\neg B>),$$

with $d\llbracket z{:}\neg E\rrbracket \equiv <\lambda x_0{:}F.e_1\llbracket x_0,z\rrbracket{:}\neg F,\ \lambda y_0{:}G.e_2\llbracket y_0,z\rrbracket{:}\neg G>$, we have $\Gamma \vdash$ (LHS) = e' : E, by $[\beta \to \lambda]$, $[\beta\gamma\bot]$, $[^h\beta\gamma \to]$, and $\Gamma \vdash$ (RHS) = e' : E, by $[\oint\gamma]$.

For $[\oint\bigvee_{\cup}\bigvee_{\natural}{:}\exists\exists]$: where

$$e' \equiv \gamma z{:}\neg E.h(<\lambda x{:}A.f\llbracket x\rrbracket(a\llbracket x\rrbracket(e_0\llbracket z\rrbracket)){:}\neg A,\ \lambda y{:}B.g\llbracket y\rrbracket(b\llbracket y\rrbracket(e_0\llbracket z\rrbracket)){:}\neg B>),$$

with $e_0\llbracket z{:}\neg E\rrbracket \equiv !u.\lambda x_0{:}F\llbracket u\rrbracket.e\llbracket u,x_0,z\rrbracket : \forall u.\neg F\llbracket u\rrbracket$, we have $\Gamma \vdash$ (LHS) = e' : E, by $[\beta \to \lambda]$, $[\beta\gamma\bot]$, and $[^h\beta\gamma \to]$, and $\Gamma \vdash$ (RHS) = e' : E, by $[\oint\gamma]$. ☐

Theorem (*Cross* $\bigvee_{\cup}$*-diagonalization in* $\lambda\gamma_{\&}\mathbf{CQ}$). For $z, z_0 \notin FV_\lambda(h,a\llbracket u,x\rrbracket,f\llbracket u,x\rrbracket)$,

$[\oint\bigvee_{\natural}\bigvee_{\cup}{:}\vee]$
$\Gamma \vdash \bigvee_{\natural}(z{:}\neg E).c \Diamond [\lambda x_0{:}F.e_1,\ \lambda y_0{:}G.e_2] = \bigvee_{\cup}(z{:}\neg E).h \Diamond [!u.\lambda x{:}A\llbracket u\rrbracket.f(z(e\llbracket u,x\rrbracket))]\ [:\ E]$,
if $\Gamma \vdash h : \exists u.A\llbracket u\rrbracket$, $\Gamma[u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket] \vdash a\llbracket u,x\rrbracket : F \vee G$, $\Gamma[u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket] \vdash f\llbracket u,x\rrbracket : \top$,
and $\Gamma[x_0{:}F][z{:}\neg E] \vdash e_1\llbracket x_0,z\rrbracket : \bot$, $\Gamma[y_0{:}G][z{:}\neg E] \vdash e_2\llbracket y_0,z\rrbracket : \bot$, $[u \notin FV_u(E,e_1\llbracket x_0,z\rrbracket,e_2\llbracket y_0,z\rrbracket)]$,

where
$c \equiv \bigvee_{\cup}(z_0{:}\neg C).h \Diamond [!u.\lambda x{:}A\llbracket u\rrbracket.f\llbracket u,x\rrbracket(z_0(a\llbracket u,x\rrbracket))]$, for $C \equiv F \vee G$ $[u \notin FV_u(C)]$, and
$e\llbracket u,x\rrbracket \equiv \bigvee_{\natural}(z{:}\neg E).a\llbracket u,x\rrbracket \Diamond [\lambda x_0{:}F.e_1\llbracket x_0,z\rrbracket,\ \lambda y_0{:}G.e_2\llbracket y_0,z\rrbracket]$,

$[\oint\bigvee_{\cup}\bigvee_{\cup}{:}\exists]$
$\Gamma \vdash \bigvee_{\cup}(z{:}\neg E).c \Diamond [!v.\lambda x_0{:}F\llbracket v\rrbracket.e] = \bigvee_{\cup}(z{:}\neg E).h \Diamond [!u.\lambda x{:}A\llbracket u\rrbracket.f(z(b\llbracket u,x\rrbracket))]\ [:\ E]$,
if $\Gamma \vdash h : \exists u.A\llbracket u\rrbracket$, $\Gamma[u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket] \vdash a\llbracket u,x\rrbracket : \exists v.F\llbracket v\rrbracket$, $\Gamma[u{:}\mathbf{U}][x{:}A\llbracket u\rrbracket] \vdash f\llbracket u,x\rrbracket : \top$,
and $\Gamma[v{:}\mathbf{U}][x_0{:}F\llbracket v\rrbracket][z{:}\neg E] \vdash e\llbracket v,x_0,z\rrbracket : \bot$, $[u \notin FV_u(e\llbracket v,x_0,z\rrbracket)]$, $[v \notin FV_u(E)]$,

where
$c \equiv \bigvee_{\cup}(z_0{:}\neg C).h \Diamond [!u.\lambda x{:}A\llbracket u\rrbracket.f\llbracket u,x\rrbracket(z_0(a\llbracket u,x\rrbracket))]$, for $C \equiv \exists v.F\llbracket v\rrbracket$, $[u \notin FV_u(C)]$, and
$b\llbracket u,x\rrbracket \equiv \bigvee_{\cup}(z{:}\neg E).a\llbracket u,x\rrbracket \Diamond [!v.\lambda x_0{:}F\llbracket v\rrbracket.e\llbracket v,x_0,z\rrbracket]$.

Proof. Routine [*exercise*]. The minimal derivability conditions in $\lambda\gamma_{\&}\mathbf{CQ}$ are as follows.

For $[\oint\bigvee_{\natural}\bigvee_{\cup}{:}\vee]$: where

$e' \equiv \gamma z{:}\neg E.h(!u.\lambda x{:}A\llbracket u\rrbracket.f\llbracket u,x\rrbracket(a\llbracket u,x\rrbracket(d\llbracket z\rrbracket)))$, with $d\llbracket z{:}\neg E\rrbracket \equiv <\lambda x_0{:}F.e_1\llbracket x_0,z\rrbracket{:}\neg F,\ \lambda y_0{:}G.e_2\llbracket y_0,z\rrbracket{:}\neg G>$,

one has $\Gamma \vdash$ (LHS) = e' : E, by $[\beta \to \lambda]$, $[\beta\gamma\bot]$, and $[^h\beta\gamma \to]$, and $\Gamma \vdash$ (RHS) = e' : E, by $[\oint\gamma]$.

For $[\oint\bigvee_{\cup}\bigvee_{\cup}{:}\exists]$: where

$e' \equiv \gamma z{:}\neg E.h(!u.\lambda x{:}A\llbracket u\rrbracket.f\llbracket u,x\rrbracket(a\llbracket u,x\rrbracket(d\llbracket z\rrbracket)))$, with $d\llbracket z{:}\neg E\rrbracket \equiv !v.\lambda x_0{:}F\llbracket v\rrbracket.e\llbracket v,x_0,z\rrbracket : \forall v.\neg F\llbracket v\rrbracket$,

one has $\Gamma \vdash$ (LHS) = e' : E, by $[\beta \to \lambda]$, $[\beta\gamma\bot]$, and $[^h\beta\gamma\forall]$, and $\Gamma \vdash$ (RHS) = e' : E, by $[\oint\gamma]$. ☐

Remark ("$[\bigvee_{\natural},\bigvee_{\cup}]$-*diagonal situations*": $\lambda\gamma_{\&}\mathbf{CQ}$*-derivability conditions*). The derivations of the Boolean "$[\bigvee_{\natural},\bigvee_{\cup}]$-diagonal" rules in $\lambda\gamma_{\&}\mathbf{CQ}$ do not depend on the "extensionality" conditions $[\eta\wedge]$ and $[\eta\forall]$. However, as expected, these derivations *require diagonalization* $[\oint\gamma]$.

As shown below, [$\oint\bigvee_{\natural}\bigvee_{\natural}$:$\vee\vee$] and [$\oint\bigvee_{\cup}\bigvee_{\natural}$:$\exists\exists$] specialize to appropriate **HQ** "$\vee$-commuting" rules, whereas [$\oint\bigvee_{\natural}\bigvee_{\cup}$:$\vee$], [$\oint\bigvee_{\cup}\bigvee_{\cup}$:$\exists$] specialize to **HQ** "$\exists$-commuting" rules. These are the "negative Heyting [$\vee$,$\exists$]-commuting rules".

Like its first half, the following *exercise* is meant to evidentiate specific features of the *non-Brouwerian* contents of **CQ**.

Exercise (*Negative* **DQ**-*proof operators*: *second part*). State and derive, in $\lambda\gamma_{\&}$**CQ**, appropriate [$\vee$,$\exists$]$_d$-rules for the Curry logic **DQ** of "complete refutability", in analogy with the **CQ**-rules reviewed above, viz.,

- $\beta\eta$-[$\vee$,$\exists$]$_d$-rules,
- *positive* β-[$\sqcup_d$,$\amalg_d$]-rules, and
- "*commuting*" [$\oint\sqcup_d$]- and [$\oint\amalg_d$]-rules (i.e., *cross* [$\sqcup_d$,$\amalg_d$]-*diagonal situations*),

using the Boolean simulations of $\mathbf{j}^1$, $\mathbf{j}^2$, $\mathbf{J}_d$, $\sqcup_d$, $\amalg_d$ resp., suggested in the discussion of proof-[term]-syntax.

Chapter VI

Negative inferential proof-operators: γ, ε and ω

Splitting γ: *ex falso quodlibet and consequentia mirabilis*. For subsequent purposes, it is appropriate to examine succintly the $\lambda\gamma_{\&}\mathbf{CQ}$-proof-properties of two *complementary* instances of the γ-abstractions (*reductio ad absurdum*). These are well-known in the (pre-Fregean) logical tradition as *ex falso quodlibet* and *consequentia mirabilis* [the *Law of Clavius*], resp.

(1) *Ex falso quodlibet*. Within the extended structure $\vdash_*[\mathbf{CQ}]$, the definition of $\omega_{\mathrm{A}}(\mathrm{a}{:}\bot)$ [$\equiv \omega(\mathrm{a}{:}\bot){:}\mathrm{A} := \gamma\mathrm{x}{:}\neg\mathrm{A.a},\ \mathrm{x} \notin \mathrm{FV}_\lambda(\mathrm{a})$] and the $\omega[\![\mathrm{A}]\!]$-combinator family ($\Vdash \mathrm{A} :: \mathbf{H}$, A in $[\bot,\to,\wedge,\vee,\forall,\exists]$) yield immediately, by the "cut" rule $< \$K >$, the $\vdash_*[\mathbf{CQ}]$-image of the familiar "*falsum*-rule" of Heyting's logic:

Lemma (*The "falsum"-rule*, *ex falso quodlibet*). For all formulas A of $[\bot,\to,(\wedge,\vee,\forall,\exists)]$, and any "**U**-context" Γ_u containing the **U**-parameters of $\mathrm{FV}_u(\mathrm{A})$,

$(\to\mathrm{i}\omega)_h$ $\Gamma \vdash \mathrm{a} : \bot \Rightarrow \Gamma \vdash \omega_{\mathrm{A}}(\mathrm{a}) : \mathrm{A}$ [*the "falsum"-rule*],
(ω): $\Gamma_u \vdash \omega[\![\mathrm{A}]\!] : \bot \to \mathrm{A}$ [*ex falso quodlibet*].

Clearly, $(\to\mathrm{i}\omega)_h$ obtains in the minimal Boolean structure $\vdash[\mathbf{CQ}]$. There are several ways of generalizing this operator (shown below).

(2) *Consequentia mirabilis*, *the "laws" of Clavius* (1574) *and Peirce* (1885). The classical tautologies $\mathrm{A} \to \mathrm{B} \to \mathrm{A} \to \mathrm{A}$ and (its instance) $\neg\mathrm{A} \to \mathrm{A} \to \mathrm{A}$ have a venerable history (scattered references can be found below). We examine corresponding proofs-forms (proof-operators, proof-combinators). For all formulas A, B in $[\bot,\to,(\wedge,\vee,\forall,\exists)]$, one has:

Definition (*The Rule of Clavius and the "Clavian functionals"*).

$\varepsilon\mathrm{x}{:}\neg\mathrm{A.a}[\![\mathrm{x}]\!]$ [$\equiv \varepsilon([\mathrm{x}{:}\neg\mathrm{A}].\mathrm{a}[\![\mathrm{x}]\!]{:}\mathrm{A}){:}\mathrm{A}$] $:= \gamma\mathrm{x}{:}\neg\mathrm{A.x}(\mathrm{a}[\![\mathrm{x}]\!])$,
$\mathbf{E}[\![\mathrm{A}]\!] := \lambda\mathrm{x}{:}(\neg\mathrm{A}\to\mathrm{A}).\varepsilon\mathrm{y}{:}\neg\mathrm{A.x(y)}$ [$\equiv \lambda\mathrm{x}{:}(\neg\mathrm{A}\to\mathrm{A}).\gamma\mathrm{y}{:}\neg\mathrm{A.y(x(y))}$].

Definition (*The Rule of Peirce and Peirce's Law*).

$\epsilon_{\mathrm{A,B}}(\mathrm{f})$ [$\equiv \epsilon(\mathrm{f}{:}\mathrm{A}\to\mathrm{B}\to\mathrm{A}){:}\mathrm{A}$] $:= \varepsilon\mathrm{x}{:}\neg\mathrm{A.f}(\lambda\mathrm{y}{:}\mathrm{A}.\omega_{\mathrm{B}}(\mathrm{x(y)}))$ [$\mathrm{x} \notin \mathrm{FV}_\lambda(\mathrm{f})$],
$\epsilon[\![\mathrm{A,B}]\!] := \lambda\mathrm{x}{:}(\mathrm{A}\to\mathrm{B}\to\mathrm{A}).\epsilon_{\mathrm{A,B}}(\mathrm{x})$ [$\equiv \lambda\mathrm{x}{:}(\mathrm{A}\to\mathrm{B}\to\mathrm{A}).\varepsilon\mathrm{y}{:}\neg\mathrm{A.x}(\lambda\mathrm{z}{:}\mathrm{A}.\omega_{\mathrm{B}}(\mathrm{y(z)}))$]
[$\equiv \lambda\mathrm{x}{:}(\mathrm{A}\to\mathrm{B}\to\mathrm{A}).\gamma\mathrm{y}{:}\neg\mathrm{A.y}(\mathrm{x}(\lambda\mathrm{z}{:}\mathrm{A}.\gamma\mathrm{z}_0{:}\neg\mathrm{B.y(z)}))$].

Lemma (*The "Law of Clavius" and "Peirce's Law"*). For all formulas A, B in $[\bot,\to,(\wedge,\vee,\forall,\exists)]$, and any proof-context Γ,

$(\to\mathrm{i}\varepsilon)$ $\Gamma[\mathrm{x}{:}\neg\mathrm{A}] \vdash \mathrm{c}[\![\mathrm{x}]\!] : \mathrm{A} \Rightarrow \Gamma \vdash \varepsilon\mathrm{x}{:}\neg\mathrm{A.c}[\![\mathrm{x}]\!] : \mathrm{A}$ [*Clavius' Rule*],
$(\to\mathrm{i}\epsilon)$ $\Gamma \vdash \mathrm{f} : \mathrm{A} \to \mathrm{B} \to \mathrm{A} \Rightarrow \Gamma \vdash \epsilon_{\mathrm{A,B}}(\mathrm{f}) : \mathrm{A}$ [*Peirce's Rule*],

whence also, in any "**U**-context" Γ_u containing the **U**-parameters of $\mathrm{FV}_u(\mathrm{A})$, $\mathrm{FV}_u(\mathrm{A,B})$, resp.

(**E**) $\Gamma_u \vdash \mathbf{E}[\![\mathrm{A}]\!] : \neg\mathrm{A} \to \mathrm{A} \to \mathrm{A}$, [*the Law of Clavius*],
(ϵ) $\Gamma_u \vdash \epsilon[\![\mathrm{A,B}]\!] : \mathrm{A} \to \mathrm{B} \to \mathrm{A} \to \mathrm{A}$, [*Peirce's Law*].

Proof [*exercise*]. ☐

Remark (*The "Law of Clavius" and Curry's logic* **DQ**). From a provability-only point of view, the so-called "*Law of Clavius*" $\neg\mathrm{A} \to \mathrm{A} \to \mathrm{A}$ (also known as *consequentia mirabilis*), is an instance of Peirce's Law ($\mathrm{A} \to \mathrm{B} \to \mathrm{A} \to \mathrm{A}$, with $\bot$ for B). Proof-theoretically, the "*Clavian functionals*" $\mathbf{E}[\![\mathrm{A}]\!]$ (so called here following a post-Rennaissance [Jesuit] tradition, after Christoph Klau SJ [*Lat. Clavius*]) can be shown to be *equal to* $\epsilon[\![\mathrm{A},\bot]\!]$, within the least Boolean equational theory $\lambda\gamma(!)$ on $\vdash[\mathbf{CQ}]$ (cf. [Rezuş 90]: this is, essentially, $\lambda\gamma_{\&}\mathbf{CQ}$ without diagonalization, $\wedge$-types and $\wedge$-proof-primitives). Indeed, for any "**U**-context" Γ_u containing the **U**-parameters of $\mathrm{FV}_u(\mathrm{A})$,

$$\begin{aligned}\Gamma_u \vdash \epsilon[\![A,\bot]\!] &\equiv \lambda x{:}(\neg A{\to}A).\gamma y{:}\neg A.y(x(\lambda z{:}A.\gamma z_0{:}\top.y(z)))\\ &\equiv \lambda x{:}(\neg A{\to}A).\varepsilon y{:}\neg A.x(\lambda z{:}A.\omega_\bot(y(z)))\\ &= \lambda x{:}(\neg A{\to}A).\varepsilon y{:}\neg A.x(\lambda z{:}A.y(z))\\ &= \lambda x{:}(\neg A{\to}A).\varepsilon y{:}\neg A.x(y) \equiv \mathbf{E}[\![A]\!] : A \to \bot \to A \to A,\end{aligned}$$

by $[\beta\gamma\bot]$, $[\eta \to \lambda]$, resp., whence, in view of [**UT**], we have $\Gamma_u \vdash \mathbf{E}[\![A]\!] : \neg A \to A \to A$. In general, one has

$$\Gamma[f{:}(\neg A{\to}A)] \vdash \varepsilon x{:}\neg A.f(x) = \mathbf{E}[\![A]\!](f) : A.$$

So, we can use the combinator-family $\mathbf{E}[\![A]\!]$ in place of the ε-abstractor.

The proof-systems based on at least $[\bot,\to]$ and the inferential proof-operators $\lambda_\vdash$, $@_\vdash$ (or $\bigwedge_\vdash$) and ε (i.e., λ-abstraction, "functional" application and "Clavian" abstraction ε, in place of the Boolean γ-abstraction) record the inferential proof-structure of Curry's [52,63] logic of "complete refutability" **DQ**. In fact, the three proof-operators above suffice only for the $[\to,\neg]$-fragment of **DQ**, with $\neg_d$ ("strict negation", in Curry's terms) defined inferentially from $[\bot_d,\to_d]$. The corresponding "Clavian" proof-theories $\lambda\varepsilon!$, $\lambda\varepsilon\oint_0!$, etc. defined on $\vdash$[**CQ**] (and/or the extended structures $\vdash_{(*,\&)}$[**CQ**]) formalize the equational proof-behavior of (appropriate fragments of) **DQ**. On the full **DQ**, see also [Seldin 89]. For the early history of the "Law of Clavius" [also known as *consequentia mirabilis*], see [Clavius 1611] **1.1**, pp. 364–365, *ad* Eucl. **IX.12** and **1.2**, page 11, *ad* Theodos. **I.12**, [Saccheri 1697,1733] *passim*, and, possibly, [Cardano 1663] **4**, page 579 [= *De proportionibus*, Lib. **V**, Prop. 201].[22]For the "Law of Peirce", see [Peirce 1885] (Peirce's fifth "icon").

(3) *The "Clavian" ε-abstractions*. The basic equational properties (in $\lambda\gamma_\&$**CQ**) of the "Clavian" ε-abstractor are as follows.

Theorem (*Basic ε-properties in* $\lambda\gamma_\&$**CQ**). For all formulas A, B in $[\bot,\to,(\wedge,\forall)]$, and any proof-context Γ,

$[\beta\varepsilon\bot]$ $\Gamma \vdash \varepsilon x{:}\top.e[\![x]\!] = e[\![x{:=}\Omega]\!] : \bot$, if $\Gamma[x{:}\top] \vdash e[\![x]\!] : \bot$,
$[\beta\varepsilon\to]$ $\Gamma \vdash (\varepsilon x{:}\neg(A{\to}B).f[\![x]\!])(a) = \varepsilon x{:}\neg B.f[\![x{:=}\lambda z{:}(A{\to}B).x(z(a))]\!](a) : B$,
if $\Gamma[x{:}\neg(A{\to}B)] \vdash f[\![x]\!] : A \to B$, $\Gamma \vdash a : A$,

[22] The "Clavian functionals" could have been better called "*E*uclidean proof-operators", since they occur already in *Elementa*, IX.12, This has been noticed first by Christoph Clavius SJ *locc. citt.*. Clavius acted his entire life as a professor of mathematics (in the Jesuit's *Collegium Romanum*) and was not particularly interested in logic matters (then "dialectic") as a stand-alone subject, although, without any doubt, he did always keep an alert eye on the use of specific inferences in Euclid's *Elements*; in retrospect, we could qualify best this intellectual attitude as a "concern with applied logic" perhaps. The somewhat improper label "Law of Clavius" – implying Clavius as a would-be "discoverer" of this figure of proof – perpetrates later into logic books *via* a loose way of speaking used in Jesuit learned media of the XVIth and XVIIth centuries (mathematicians and theologians). The *mirabilis consequentia* – first called so likely by Gerolamo Cardano – deserve special attention in the works of Gerolamo Saccheri SJ, referred to *supra*: in fact, the Genoan *vindicator* of Euclid did, indirectly, spend a life-time on it… (For more information, see, e.g., [Vailati 03,03a,04,04a].) Among the moderns, Haskell B. Curry has been the first (± 1950) to make relevant use of the "Clavian" combinators and the derived ε-operations. Curry's logic of "strict negation" remained, practically, a technical *curiosity*, as a way of playing with negation: the outcome is *purely inferential*, in contrast with various notions of negation extracted from "non-classical" semantical considerations (on many-valuedness, truth-value "gaps", and the like). The least we can say is that the "strict" negation is *non-Brouwerian*, although this is not very illuminating. It is also unclear in what sense should we call it "strict", as Curry used to think of it. Within the full classical setting, discussed here, a tempting way of understanding Curry's idea, would consist of the observation that *the inferential part of this logic does not allow "cancelling" γ's*, exactly in the same sense the λ**I**-calculus of Alonzo Church [41] – and so the inferential part of a well-known "relevant" logic, i.e., Church's "theory of weak implication" – does not allow "cancelling" λ's. Such a hypothesis would be, however, in conflict with Curry's "reductionist" view on abstraction: he was convinced of the fact that *every* kind of abstraction can be defined explicitly in terms of λ-abstraction and additional functionals. (An immediate consequence of this tenet is that one *cannot* correctly explain classical logic, *qua* proof-behavior. The genuine γ-operators are *abstractors*, in general *not* "reducible" to λ's. The use of Δ's to this purpose – i.e., double-negation eliminations; see the *exercises* following below or [Rezuş 90] – is circular, of course.) So, the "reductionist" view on abstraction-operators should have rendered Curry unable to distinguish among "cancelling" λ's (allowed in his logic) and "non-cancelling" γ's (not allowed there). As puzzling as it appears, the logic of "strict negation" ["complete refutability"] is *correctly formulated as an inferential logic*, even if its main proponent and his forerunners didn't have the right tools to think about it…

$[\beta\varepsilon\wedge_1]$ $\Gamma \vdash \mathbf{p}_1(\varepsilon x{:}\neg(A\wedge B).f[\![x]\!]) = \varepsilon x{:}\neg A.\mathbf{p}_1(f[\![x{:=}\lambda z{:}(A\wedge B).x(\mathbf{p}_1(z))]\!]) : A$,
if $\Gamma[x{:}\neg(A\wedge B)] \vdash f[\![x]\!] : A \wedge B$,
$[\beta\varepsilon\wedge_2]$ $\Gamma \vdash \mathbf{p}_2(\varepsilon x{:}\neg(A\wedge B).f[\![x]\!]) = \varepsilon x{:}\neg B.\mathbf{p}_2(f[\![x{:=}\lambda z{:}(A\wedge B).x(\mathbf{p}_2(z))]\!]) : B$,
if $\Gamma[x{:}\neg(A\wedge B)] \vdash f[\![x]\!] : A \wedge B$,
$[\beta\varepsilon\forall]$ $\Gamma \vdash (\varepsilon x{:}\neg(\forall u.A[\![u]\!]).f[\![x]\!])[\mathbf{t}] = \varepsilon x{:}\neg A[\![u{:=}\mathbf{t}]\!].f[\![x{:=}\lambda z{:}(\forall u.A[\![u]\!]).x(z[\mathbf{t}])]\!][\mathbf{t}] : A[\![u{:=}\mathbf{t}]\!]$,
if $\Gamma[x{:}\neg(\forall u.A[\![u]\!])] \vdash f[\![x]\!] : \forall u.A[\![u]\!]$, $\Gamma \Vdash \mathbf{t} :: \mathbf{U}$,
$[\eta{\to}\varepsilon]$ $\Gamma \vdash \varepsilon x{:}\neg A.a = a : A$, if $\Gamma \vdash a : A$, $[x \notin FV_\lambda(a)]$,
$[\oint_0\varepsilon]$ $\Gamma \vdash \varepsilon x{:}\neg A.\varepsilon y{:}\neg A.a[\![x,y]\!]) = \varepsilon z{:}\neg A.a[\![x{:=}z]\!][\![y{:=}z]\!] : A$, if $\Gamma[x{:}\neg A][y{:}\neg A] \vdash a[\![x,y]\!] : A$,
$[\varepsilon\omega]$ $\Gamma \vdash \varepsilon x{:}\neg A.\omega_A(x(a)) = a : A$, if $\Gamma \vdash a : A$, $[x \notin FV_\lambda(a)]$.

Proof. Easy calculations [*exercise*]. E.g., $[\beta\varepsilon{\to}]$, follows from $[\beta\gamma \to]$, and $[\beta \to \lambda]$. □

Remark (*ε-congruence*). In view of $[\xi \to \gamma]$ and $[\nu \to]$, one has also immediately, for all formulas A in $[\bot,\to,(\wedge,\forall)]$, and any proof-context Γ,

$[\xi{\to}\varepsilon]$ $\Gamma \vdash \varepsilon x{:}\neg A.a_1[\![x]\!] = \varepsilon x{:}\neg A.e_2[\![x]\!]$ [: A], if $\Gamma[x{:}\neg A] \vdash a_1[\![x]\!] = a_2[\![x]\!] : A$

Without $[\varepsilon\omega]$, the ε-properties above, yield purely "Clavian" sub-theories $\lambda\varepsilon(\pi)!$, $\lambda\varepsilon\oint_0(\pi)!$ of $\lambda\gamma_{(\&)}\mathbf{CQ}$ matching the $[\bot_d,\to_d,(\wedge_d),\forall_d]$-fragment(s) of **DQ**. The minimal "Clavian" theory $\lambda\varepsilon!$ has no trace of diagonalization, while $\lambda\varepsilon\oint_0(\pi)!$ has $[\oint_0\varepsilon]$. No analogue of $[\oint\gamma]$ seems to be available with ε alone.

Exercises (*"Clavian" logic: the proof-theory of* **DQ**).

(1) Conversely, in $\vdash[\mathbf{CQ}]$, the family $\mathbf{E}[\![A,B]\!]$ ($\Vdash$ A, B :: **H**), can be expressed with λ-abstraction, "functional" application and the ε-operator. Show that, for all formulas A, B in $[\bot,\to,(\wedge,\vee,\forall,\exists)]$, and any proof-context Γ,

$[\beta\mathbf{E}\bot]$ $\Gamma \vdash \mathbf{E}[\![\bot]\!](f) = f(\Omega) : \bot$, if $\Gamma \vdash f : \top \to \bot$ $[\equiv_{df} (\bot \to \bot \to \bot)]$,
$[\beta\mathbf{E}{\to}]$ $\Gamma \vdash \mathbf{E}[\![A{\to}B]\!](f)(a) = \mathbf{E}[\![B]\!](\lambda x{:}\neg B.f(c_A[\![x]\!])(a)) : B$, if $\Gamma \vdash f : \neg(A{\to}B){\to}(A{\to}B)$, $\Gamma \vdash a : A$,
where $c_A[\![x]\!] \equiv \lambda z{:}(A{\to}B).x(z(a))$,
$[\eta\mathbf{E}]$ $\Gamma \vdash \mathbf{E}[\![A]\!](\lambda x{:}\neg A.a) = a$, if $\Gamma \vdash a : A$ $[x \notin FV_\lambda(a)]$.

(2) "*Clavian functionals*" (Christoph Klau [Clavius], 1574). Derive the following **E**-equations in $\lambda\gamma_\&\mathbf{CQ}$, for all formulas A in $[\bot,\to,(\wedge,\forall)]$, where Γ_u contains the "**U**-parameters" of $FV_u(A)$:

[ext**E**] $\Gamma_u \vdash \mathbf{E}[\![A]\!] \circ \mathbf{K}_\sim[\![A]\!] = \mathbf{I}[\![A]\!]$,
[**E** Δ] $\Gamma_u \vdash \mathbf{E}[\![A]\!] = \Delta[\![A]\!] \circ \S[\![A]\!]$,
[Δ **E**] $\Gamma_u \vdash \Delta[\![A]\!] = \mathbf{E}[\![A]\!] \circ \#[\![A]\!]$,

where ∘ stands, in each case, for the appropriate stratified **B**-combinator (composition), and

$\mathbf{K}_\sim[\![A]\!] := \mathbf{K}[\![A,\neg A]\!]$ $[\equiv \lambda x{:}A.\lambda y.\neg A.x]$,
$\Delta[\![A]\!]$ $[\equiv \lambda x{:}\neg\neg A.\gamma y{:}\neg A.x(y)]$, as ever, and
$\#[\![A]\!] := \dagger[\![\neg A,A]\!]$ $[\equiv \lambda x{:}\neg\neg A.\lambda y{:}\neg A.\omega_A(x(y))]$,
$\S[\![A]\!] := \mathbf{S}_\star[\![A,\bot]\!]$ $[\equiv \lambda x{:}(\neg A{\to}A).\lambda y{:}(\neg A).y(x(y))]$,

with $\Gamma_u \vdash \#[\![A]\!] : \neg\neg A \to \neg A \to A$ and $\Gamma_u \vdash \S[\![A]\!] : \neg A \to A \to \neg\neg A$.

Show that diagonalization $[\oint\gamma]$ is not actually needed.

(3) Describe, on the pattern of the present notes, the proof-theory of the logic of "complete refutability" **DQ** as a typed λ-calculus, λ**DQ**, say. (*Hint*. Cf. [Curry 52,63], [Seldin 89] for provability matters, and recall the previous *extended exercises* on the "negative" **DQ**-proof-operators.)

(4) *Deferred inferential proof-operators*. The γ-abstractions can be analyzed in terms of ω-operators, required in **HQ**, and "Clavian" ε-abstractions [= applications of the *Rule of Clavius*], required in **DQ**. Indeed, it is easy to see that the stratification rules $({\to}i\varepsilon)$ and $({\to}i\omega)$ are tantamount $({\to}i\gamma)$: using the former two rules, in place of $({\to}i\gamma)$, one could have simulated the stratification-behavior of the γ-abstractor by setting, e.g.,

γ°x:¬A.e⟦x⟧ := εx:¬A.ω⟦A⟧(e⟦x⟧) [$\equiv_{df}$ εx:¬A.ω_{A}(e⟦x⟧)].

One can generalize this as follows.

Definition (*Deferred inferential proof-operators*).

For all formulas A in [⊥,→,(∧,∨,∀,∃)],

γ°x:¬A.e⟦x⟧ [≡ γ°([x:¬A].e⟦x⟧:⊥):A] := εx:¬A.ω_{A}(e⟦x⟧),
$\omega^\circ_{\mathrm{A}}$(e) [≡ ω°(e:⊥):A] := γ°x:¬A.e [x ∉ FV_λ(e)],
ε°x:¬A.a⟦x⟧ [≡ ε°([x:¬A].a⟦x⟧:A):A] := γ°x:¬A.x(a⟦x⟧).

In general, set, for $n \geq 0$, and all formulas A in [⊥,→,(∧,∨,∀,∃)],

$\gamma^{[0]}$x:¬A.e⟦x⟧ [≡ $\gamma^{[0]}$([x:¬A].e⟦x⟧:⊥):A] := γx:¬A.e⟦x⟧,
$\gamma^{[n+1]}$x:¬A.e⟦x⟧ [≡ $\gamma^{[n+1]}$([x:¬A].e⟦x⟧:⊥):A] := $\varepsilon^{[n]}$x:¬A.$\omega^{[n]}_{\mathrm{A}}$(e⟦x⟧), **where**
$\omega^{[n]}_{\mathrm{A}}$(e) [≡ $\omega^{[n]}_{\mathrm{A}}$(e:⊥):A] := $\gamma^{[n]}$x:¬A.e, [x ∉ FV_λ(e)],
$\varepsilon^{[n]}$x:¬A.a⟦x⟧ [≡ $\varepsilon^{[n]}$([x:¬A].a⟦x⟧:A):A] := $\gamma^{[n]}$x:¬A.x(a⟦x⟧).

Lemma (*Deferred inferential proof-operators: stratification*). For all formulas A in [⊥,→,(∧,∨,∀,∃)], and any proof-context Γ,

($\to i\gamma^\circ$) Γ[x:¬A] ⊢ e⟦x⟧ : ⊥ ⇒ Γ ⊢ γ°x:¬A.e⟦x⟧ : A,
($\to i\omega^\circ$) Γ ⊢ e : ⊥ ⇒ Γ ⊢ $\omega^\circ_{\mathrm{A}}$(e) : A,
($\to i\varepsilon^\circ$) Γ[x:¬A] ⊢ a⟦x⟧ : A ⇒ Γ ⊢ ε°x:¬A.a⟦x⟧ : A,
and, in general, for all $n \geq 0$,
($\to i\gamma^{[n]}$) Γ[x:¬A] ⊢ e⟦x⟧ : ⊥ ⇒ Γ ⊢ $\gamma^{[n]}$x:¬A.e⟦x⟧ : A,
($\to i\omega^{[n]}$) Γ ⊢ e : ⊥ ⇒ Γ ⊢ $\omega^{[n]}_{\mathrm{A}}$(e) : A,
($\to i\varepsilon^{[n]}$) Γ[x:¬A] ⊢ a⟦x⟧ : A ⇒ Γ ⊢ $\varepsilon^{[n]}$x:¬A.a⟦x⟧ : A.

Proof. ($\to i\gamma^\circ$): use ($\to i\omega$) and ($\to i\varepsilon$). ($\to i\omega^\circ$), ($\to i\varepsilon^\circ$): equally trivial. ($\to i\gamma^{[n]}$): One can show, in fact, that, for all $n \geq 0$,

($\to i\gamma^{[n+1]}$) Γ ⊢ $\gamma^{[n+1]}$x:¬A.e⟦x⟧ ≡ $\varepsilon^{[n]}$x:¬A.$\omega^{[n]}_{\mathrm{A}}$(e⟦x⟧) ≡ $\gamma^{[n]}$x:¬A.x($\gamma^{[n]}$y:¬A.e⟦x⟧) : A
if Γ[x:¬A] ⊢ e⟦x⟧ : ⊥ [y ∉ FV_λ(c⟦x⟧)],

whence ($\to i\gamma^{[n]}$) follows by induction on n. ($\to i\omega^{[n]}$), ($\to i\varepsilon^{[n]}$): These are special cases of ($\to i\gamma^{[n]}$). □

Within $\lambda\gamma_{\&}$**CQ**, the "simply deferred" γ°-abstractor γ°x:¬A.e⟦x⟧ has the same equational behavior as the primitive γ-abstractor.

Lemma (*Simply deferred γ-abstraction: equational behavior*). For all A in [⊥,→,(∧,∀)], one has, in $\lambda\gamma_{\&}$**CQ**,

[$\gamma^\circ\gamma$] Γ[x:¬A] ⊢ e⟦x⟧ : ⊥ ⇒ Γ ⊢ γ°x:¬A.e⟦x⟧ = γx:¬A.e⟦x⟧ : A,

Proof. Obvious (use [$\oint_0\gamma$]). □

Analogously, "deferring once" ω, ε yields proof-operators ω°, ε° resp. with the same equational properties.

Corollary (*Simply deferred negative inferential proof-operators*). For all A in [⊥,→,(∧,∀)], and any proof-context Γ, one has, in $\lambda\gamma_{\&}$**CQ**,

[$\omega^\circ\omega$] Γ ⊢ e : ⊥ ⇒ Γ ⊢ $\omega^\circ_{\mathrm{A}}$(e) = ω_{A}(e) : A,
[$\varepsilon^\circ\varepsilon$] Γ[x:¬A] ⊢ a⟦x⟧ : A ⇒ Γ ⊢ ε°x:¬A.a⟦x⟧ = εx:¬A.a⟦x⟧ : A.

Proof. Easy [*exercise*] (use [$\oint_0\gamma$]). □

We have already noticed the fact that the *deferring* operation can be generalized in a straightforward way such as to obtain a $\gamma^{[n]}$-$\varepsilon^{[n]}$-$\omega^{[n]}$-sequence ($n \geq 0$). The diagonalization rule [$\oint\gamma$] of $\lambda\gamma_{(\&)}$**CQ** – actually, just the weak diagonalization [$\oint_0\gamma$] – induces a *reflection property* on the equational behavior of γ-proof-operators, insuring the fact that the $\gamma^{[n]}$-$\varepsilon^{[n]}$-$\omega^{[n]}$-sequence ($n \geq 0$) behaves, in $\lambda\gamma_{(\&)}$**CQ**, like the γ-ε-ω-triple.

Theorem (*Deferred negative proof-operators: equational behavior*). In general, for all $n \geq 0$, one has, in $\lambda\gamma_{(\&)}\mathbf{CQ}$, for all formulas A in [⊥,→,(∧,∀)], and any proof-context Γ,

$[\gamma^{[n]}\gamma]$ $\Gamma \vdash \gamma^{[n]}x{:}\neg A.e[\![x]\!] = \gamma x{:}\neg A.e[\![x]\!]{:}\ A$, if $\Gamma[x{:}\neg A] \vdash e[\![x]\!] : \bot$,

$[\omega^{[n]}\omega]$ $\Gamma \vdash \omega^{[n]}_{A}(e) = \omega_A(e) : A$, if $\Gamma \vdash e : \bot$,

$[\varepsilon^{[n]}\varepsilon]$ $\Gamma \vdash \varepsilon^{[n]}x{:}\neg A.a[\![x]\!] = \varepsilon x{:}\neg A.a[\![x]\!] : A$, if $\Gamma[x{:}\neg A] \vdash a[\![x]\!] : A$.

Proof. Routine [*exercise*]. (*Hint*. One needs $[\oint_0\gamma]$.) □

We are now ready to go into the proper (equational) behavior of ω (and so into the *idiosyncrasies* of the Heyting proofs).

Boolean and Heyting ω-rules in $\lambda\gamma_{\&}\mathbf{CQ}$. As mentioned earlier, in contrast with the first-order fragment of Martin-Löf's type theory and the *Minimalkalkül*, Heyting's first-order proof-calculus has a number of *ad hoc*-looking rules (the so-called "⊥-rules", here: *ω-rules* or *ex-falso*-rules), describing the specific equational/reduction properties of the ω-operator. These rules characterize the behavior of the ω-proof-terms of the form $\omega_C(e{:}\bot)$ [for C := ⊥, (A→B), (A∧B), (A∨B), (∀u.A⟦u⟧) or (∃u.A⟦u⟧)], occurring within the scope of a selector (or within the scope of ω [**HQ** thinks of ω as being a selector]). (For "natural deduction" variants, see [Prawitz 65], [Troelstra 73], [Troelstra & van Dalen 88].)

Since, in a classical proof-setting, the ω-operator is a specific instance of γ-abstraction, one might suspect that the ω-rules are available in $\lambda\gamma_{\&}\mathbf{CQ}$. This is, indeed, the case. In fact, in $\lambda\gamma_{\&}\mathbf{CQ}$, one has slightly more general ω-rules: specifically, this concerns the equational behavior of the ω-terms relative to the *Boolean* [∨,∃]-proof-operators.

Theorem (*The Boolean ω-rules in* $\lambda\gamma_{\&}\mathbf{CQ}$). For all A, B, C in [⊥,→,∧,∀], and any proof-context Γ,

$[\beta\omega\bot] \equiv [\eta\bot\omega]_h$ $\Gamma \vdash \omega_\bot(f) = f\ [{:}\ \bot]$, if $\Gamma \vdash f : \bot$,

$[\beta\omega\to]$ $\Gamma \vdash (\omega_F(f))(a) = \omega_B(f)\ [{:}\ B]$, if $\Gamma \vdash f : \bot$, $\Gamma \vdash a : A$, $[F \equiv A \to B]$,

$[\beta\omega\wedge_1]$ $\Gamma \vdash \mathbf{p}_1(\omega_F(f)) = \omega_A(f)\ [{:}\ A]$, if $\Gamma \vdash f : \bot$, $[F \equiv A \wedge B]$,

$[\beta\omega\wedge_2]$ $\Gamma \vdash \mathbf{p}_2(\omega_F(f)) = \omega_B(f)\ [{:}\ B]$, if $\Gamma \vdash f : \bot$, $[F \equiv A \wedge B]$,

$[\beta\omega\vee]$ $\Gamma \vdash \bigvee_\natural(z{:}\neg C).\omega_F(f) \diamondsuit [\lambda x{:}A.e_1[\![x,z]\!], \lambda y{:}B.e_2[\![y,z]\!]] = \omega_C(f)\ [{:}\ C]$,
if $\Gamma \vdash f : \bot$, $\Gamma[x{:}A][z{:}\neg C] \vdash e_1[\![x,z]\!] : \bot$, $\Gamma[y{:}B][z{:}\neg C] \vdash e_2[\![y,z]\!] : \bot$, $[F \equiv A \vee B]$,

$[\beta\omega\forall]$ $\Gamma \vdash (\omega_F(f))[\mathbf{t}] = \omega_G(f)\ [{:}\ A[\![u{:=}\mathbf{t}]\!]]$, if $\Gamma \vdash f : \bot$, $[\Gamma \Vdash \mathbf{t} :: \mathbf{U}]$, $[F \equiv \forall u.A[\![u]\!]]$, $[G \equiv A[\![u{:=}\mathbf{t}]\!]]$,

$[\beta\omega\exists]$ $\Gamma \vdash \bigvee_\cup(z{:}\neg C).\omega_F(f) \diamondsuit [!u.\lambda x{:}A[\![u]\!].e[\![u,x,z]\!]] = \omega_C(f)\ [{:}\ C]$,
if $\Gamma \vdash f : \bot$, $\Gamma[u{:}\mathbf{U}][x{:}A[\![u]\!]][z{:}\neg C] \vdash e[\![u,x,z]\!] : \bot$, $[u \notin FV_u(C)]$, $[F \equiv \exists u.A[\![u]\!]]$.

Proof. $[\beta\omega\bot]$ (the alternative label $[\eta\bot\omega]_h$ is explained below): This follows from $[\beta\gamma\bot]$, since,

$$\Gamma \vdash f : \bot \Rightarrow \Gamma \vdash \omega_\bot(f) \equiv \gamma x{:}\top.f = f[\![x{:=}\Omega]\!] \equiv f, \text{ whenever } x \notin FV_\lambda(f).$$

$[\beta\omega\to]$: [*exercise*] (use $[\beta\gamma\to]$).

$[\beta\omega\wedge_1]$: If $\Gamma \vdash f : \bot$ and $F \equiv [A \wedge B]$, one has, by $[\beta\gamma\wedge_1]$,

$$\Gamma \vdash \mathbf{p}_1(\omega_F(f){:}A\wedge B) \equiv \mathbf{p}_1(\gamma z{:}\neg(A\wedge B).f) = \gamma z{:}\neg A.f \equiv \omega_A(f) : A, \text{ where } z \notin FV_\lambda(f).$$

$[\beta\omega\wedge_2]$: Analogously, using $[\beta\gamma\wedge_2]$ instead.

$[\beta\omega\vee]$: Where $F \equiv [A \vee B]$, if $\Gamma \vdash f : \bot$, $\Gamma[x{:}A][z{:}\neg C] \vdash e_1[\![x,z]\!] : \bot$, and $\Gamma[y{:}B][z{:}\neg C] \vdash e_2[\![y,z]\!] : \bot$, one has

$$\Gamma \vdash \bigvee_\natural(z{:}\neg C).\omega_F(f) \diamondsuit [\lambda x{:}A.e_1,\lambda y{:}B.e_2] = \gamma z{:}\neg C.f \equiv \omega_C(f) : C, \text{ whenever } z \notin FV_\lambda(f),$$

by $[^h\beta\gamma\to]$, $[\beta\gamma\bot]$.

$[\beta\omega\forall]$: [*exercise*] (one needs $[\beta\gamma\forall]$).

$[\beta\omega\exists]$: If $F \equiv \exists u.A[\![u]\!]$, $\Gamma \vdash f : \bot$, and $\Gamma[u{:}\mathbf{U}][x{:}A[\![u]\!]][z{:}\neg C] \vdash e[\![u,x,z]\!] : \bot$, $[u \notin FV_u(C)]$, one has

$$\Gamma \vdash \bigvee_\cup(z{:}\neg C).\omega_F(f) \diamondsuit [!u.\lambda x{:}A[\![u]\!].e) = \gamma z{:}\neg C.f \equiv \omega_C(f) : C, \text{ where } z, z_0 \notin FV_\lambda(f),$$

by $[^h\beta\gamma\to]$, $[\beta\gamma\bot]$. □

Remark (*ω-congruence in* $\lambda\gamma_{\&}$**CQ**). In view of $[\xi \rightarrow \gamma]$, one has also immediately, for all formulas A in $[\bot,\rightarrow,(\wedge,\vee,\forall,\exists)]$, and any proof-context Γ,

$[\xi{\rightarrow}\omega] \equiv [\nu \rightarrow\omega]$ $\Gamma \vdash e_1 = e_2 : \bot \Rightarrow \Gamma \vdash \omega_A(e_1) = \omega_A(e_2)$ [: A].

Remark (*The Boolean ω-rules*: $\lambda\gamma_{\&}$**CQ**-*derivability conditions*). Set, for convenience,

$[\beta\omega]_+ := \{ [\beta\omega\bot], [\beta\omega{\rightarrow}], [\beta\omega\wedge_1], [\beta\omega\wedge_2], [\beta\omega\forall] \}$, and
$[\beta\omega] := [\beta\omega]_+ \cup \{ [\beta\omega\vee], [\beta\omega\exists] \}$.

From the proof of the previous theorem, it follows that $[\oint \gamma]$, and $[\eta\wedge]$, $[\eta\forall]$ are not needed in the derivation of $[\beta\omega]$ in $\lambda\gamma_{\&}$**CQ**.

Remark (*The* [*"negative"*] *ω-rules* $[\beta\omega\vee]$, $[\beta\omega\exists]$). The ("negative") ω-rules $[\beta\omega\vee]$, $[\beta\omega\exists]$, resp. are, in fact, special cases of more general (genuinely Boolean) rules, derivable in $\lambda\gamma_{\&}$**CQ**, viz. the "main branch" $[\bigvee_{\natural}\gamma]$-, $[\bigvee_{\cup}\gamma]$-diagonalizations $[{}^h\beta\gamma\vee\text{-}\beta\vee]$, $[{}^h\beta\gamma\exists\text{-}\beta\exists]$, obtained earlier. To get the "negative" ω-rules $[\beta\omega\vee]$, $[\beta\omega\exists]$, use the "cut" $< \$K >$ in order to simplify premisses $\Gamma[x{:}\neg A][y{:}\neg B] \vdash e : \bot$, and $\Gamma[x{:}\neg A[\![\mathbf{t}]\!]] \vdash e : \bot$, whenever $x, y \notin FV_\lambda(e)$. Alternatively, $[\beta\omega\vee]$ follows from $[{}^h\beta\gamma\vee]$, and $[\beta\vee]$, while $[\beta\omega\exists]$ follows from $[{}^h\beta\gamma\exists]$, and $[\beta\exists]$ [*exercise*].

In particular, for F := ⊥, (A∨B), $(\exists u.A[\![u]\!])$, one has, as special cases, the *specific Heyting ω-rules*:

Corollary (*The Heyting "ω-diagonal" rule*). For all A in $[\bot,\rightarrow,(\wedge,\vee,\forall,\exists)]$, and any proof-context Γ,

$[\beta\omega\bot]_h$ $\Gamma \vdash \omega_A(\omega_\bot(f)) = \omega_A(f)$ [: A] if $\Gamma \vdash f : \bot$.

Proof. $[\beta\omega\bot]_h$ follows from $[\beta\omega\bot] \equiv [\eta\bot\omega]_h$. □

Corollary (*The Heyting "main branch"* $[\sqcup\omega][\amalg\omega]$-*rules in* $\lambda\gamma_{\&}$**CQ**). For all A, B, C in $[\bot,\rightarrow,\wedge,\vee,\forall,\exists]$, and any proof-context Γ,

(1) $[\beta\omega\vee]_h \equiv [{}^h\beta\omega\vee\text{-}\beta\vee]_h$:

$[\beta\omega\vee]_h$ $\Gamma \vdash \sqcup(\omega_F(f),[x{:}A].c_1[\![x]\!],[y{:}B].c_2[\![y]\!]) = \omega_C(f)$ [: C],
if $\Gamma \vdash f : \bot$, $\Gamma[x{:}A] \vdash c_1[\![x]\!] : C$, $\Gamma[y{:}B] \vdash c_2[\![y]\!] : C$, $[F \equiv A \vee B]$,

(2) $[\beta\omega\exists]_h \equiv [{}^h\beta\omega\exists\text{-}\beta\exists]_h$:

$[\beta\omega\exists]_h$ $\Gamma \vdash \amalg(\omega_F(f),[u{:}\mathbf{U}][x{:}A[\![u]\!]].c[\![u,x]\!]{:}C) = \omega_C(f)$ [: C],
if $\Gamma \vdash f : \bot$, $\Gamma[u{:}\mathbf{U}][x{:}A[\![u]\!]] \vdash c[\![u,x]\!] : C$, $[u \notin FV_u(C)]$, $[F \equiv \exists v.A[\![v]\!]]$.

Proof. $[\beta\omega\vee]_h$ $[\beta\omega\exists]_h$ are special cases of $[\beta\omega\vee]$, $[\beta\omega\exists]$,resp. That is, ultimately, *modulo* the "cut" $< \$K >$, $[{}^h\beta\omega\vee\text{-}\beta\vee]_h$ is a special case of $[{}^h\beta\gamma\vee\text{-}\beta\vee]$, while $[{}^h\beta\omega\exists\text{-}\beta\exists]_h$ is a special case of $[{}^h\beta\gamma\exists\text{-}\beta\exists]$ (cf. above the "main branch" $[\bigvee_{\natural}\gamma]$- and $[\bigvee_{\cup}\gamma]$-rules of $\lambda\gamma_{\&}$**CQ**). □

Remark (*Boolean and Heyting "intensional" ω-rules in* $\lambda\gamma$**CQ**). With $\otimes \equiv \otimes_{\sqcap}$, $\oplus \equiv \oplus_{\sqcap}$, say, and intensional algebraic $[\otimes,\oplus]$-proof-operators defined as earlier, one has also, in $\lambda\gamma$**CQ**, for all A, B, C in $[\bot,\rightarrow,(\wedge,\forall)]$, and any proof-context Γ,

$[\beta\omega\otimes_1]$ $\Gamma \vdash \pi_1(\omega_F(f){:}A{\otimes}B) = \omega_A(f) : A$, if $\Gamma \vdash f : \bot$, $[F \equiv A \otimes B]$,
$[\beta\omega\otimes_2]$ $\Gamma \vdash \pi_2(\omega_F(f){:}A{\otimes}B) = \omega_B(f) : B$, if $\Gamma \vdash f : \bot$, $[F \equiv A \otimes B]$,
$[\beta\omega\oplus]$ $\Gamma \vdash \bigvee_{\otimes}(z{:}\neg C).\omega_F(f) \diamondsuit [\lambda x{:}A.e_1,\lambda y{:}B.e_2] = \omega_C(f) : C$
if $\Gamma \vdash f : \bot$, $\Gamma[x{:}A][z{:}\neg C] \vdash e_1[\![x,z]\!] : \bot$, $\Gamma[y{:}B][z{:}\neg C] \vdash e_2[\![y,z]\!] : \bot$, $[F \equiv A \oplus B]$,

and, in particular, defining [as above],

$$\sqcup_{\otimes}(f,[x{:}A].c_1[\![x]\!],[y{:}B].c_2[\![y]\!]) \equiv_{df} \bigvee_{\otimes}(z{:}\neg C).f \diamondsuit [\lambda x{:}A.z(c_1[\![x]\!]),\lambda y{:}B.z(c_2[\![y]\!])],$$

$[z \notin FV_\lambda(c_1[\![x]\!],c_2[\![y]\!])]$, one has also the special case:

$[\beta\omega\oplus]_h$ $\Gamma \vdash \sqcup_{\otimes}(\omega_F(f),[x{:}A].c_1[\![x]\!],[y{:}B].c_2[\![y]\!]) = \omega_C(f) : C$,
if $\Gamma \vdash f : \bot$, $\Gamma[x{:}A] \vdash c_1[\![x]\!] : C$, $\Gamma[y{:}B] \vdash c_2[\![y]\!] : C$, $[F \equiv A \oplus B]$.

Proof. $[\beta\omega\otimes_1]$, $[\beta\omega\otimes_2]$, $[\beta\omega\oplus]$: Use $[\beta\to\lambda]$, $[\beta\gamma\bot]$, $[\beta\gamma\to]$. $[\beta\omega\oplus]_h$ is a special case of $[\beta\omega\oplus]$. □

Remark (*The Heyting* ω*-rules*: $\lambda\gamma_\&\mathbf{CQ}$*-derivability conditions*).

(1) In order to keep track of $\lambda\gamma_\&\mathbf{CQ}$-derivability conditions, set:

(a) $[\beta\omega]_h(\vdash_*[\mathbf{CQ}]) := \{\ [\beta\omega\bot]_h,\ [\beta\omega{\to}],\ [\beta\omega\wedge_1],\ [\beta\omega\wedge_2],\ [\beta\omega\vee]_h,\ [\beta\omega\forall],\ [\beta\omega\exists]_h\ \}$, in the primitive $\vdash_*[\mathbf{CQ}]$-syntax, resp., *mutatis mutandis*,
$[\beta\omega]_h(\vdash_\&[\mathbf{CQ}]) := [\beta\omega]_h$, in the $\vdash_\&[\mathbf{CQ}]$-syntax, and

(b) $[\beta\omega]_h(\vdash[\mathbf{CQ}]) := \{\ [\beta\omega\bot]_h, [\beta\omega{\to}], [\beta\omega\otimes_1], [\beta\omega\otimes_2], [\beta\omega\oplus]_h, [\beta\omega\forall], [\beta\omega\exists]_h\ \}$, in the $\vdash[\mathbf{CQ}]$-syntax.

In the above, only the rules $[\beta\omega\vee]$, and $[\beta\omega\exists]$ (as well as the "intensional" analogue $[\beta\omega\oplus]$ of $[\beta\omega\vee]$) are *genuinely* Boolean. On the other hand, $[\beta\omega\bot]$ can be seen as a "$\bot$-extensionality"-property, $[\eta\bot\omega]_h$ say. It obtains for Heyting's logic, too. The remaining $\beta\omega$-rules $[\beta\omega]_h(\vdash_{(*,\&)}[\mathbf{CQ}])$ are *images*, in $\lambda\gamma_\&\mathbf{CQ}$, *of the Heyting* ω*-rules*.

(2) The rules $[\beta\omega]_h(\vdash_\&[\mathbf{CQ}])$ (and so $[\beta\omega]_h(\vdash_*[\mathbf{CQ}])$] are derivable in a sub-theory ($\lambda\gamma\pi_\beta!_\beta$, say) of $\lambda\gamma_\&\mathbf{CQ}$, without diagonalization and $[\wedge,\forall]$-"extensionality" conditions. Analogously, the rules $[\beta\omega]_h(\vdash[\mathbf{CQ}])$ are derivable in a sub-theory ($\lambda\gamma!_\beta$, say) of $\lambda\gamma\mathbf{CQ}$, without diagonalization and $[\forall]$-"extensionality".

(3) Concluding, the least first-order classical theory $\lambda\gamma!$ [i.e., $\lambda\gamma\mathbf{CQ}$ *without* diagonalization] suffices in order to derive the Boolean analogues of the Heyting ω-rules. Moreover, *the Heyting* ω*-rules hold* (*classically*) *for the standard notion of proof-reduction of* $\lambda\gamma!$ [*exercise*]. (*Hint*: Cf. [Rezuş 90].)

CHAPTER VII

THE HEYTING PROOF-CALCULUS

The Heyting "commuting conversions" in $\lambda\gamma_{\&}$**CQ**. Like the "$[\bigvee_{\natural},\bigvee_{\cup}]$-diagonal" rules of $\lambda\gamma_{\&}$**CQ**, the following facts are more difficult to state than to prove. They say, in essence, that the so-called $\vee$- and $\exists$-"commuting" conversion rules of the Heyting first-order proof-calculus are available equationally in $\lambda\gamma_{\&}$**CQ**. (See, e.g., [Prawitz 65], [Troelstra 73], [Troelstra & van Dalen 88] **2**, for "natural deduction" variants.) We shall give sufficient information in order to restore full derivations: once we know what is to be proved, these rely on routine calculations in $\lambda\gamma_{\&}$**CQ**.[23]

(0) For the record, catalogued first are the special instances of $\bigvee_{\natural}$ and $\bigvee_{\cup}$, matching the γ-to-ω type of instantiation. Indeed, as in the case of the γ-abstractions, there are instances of the Boolean $[\bigvee_{\natural},\bigvee_{\cup}]$-selectors that are "intuitionistically correct", so to speak. Such instances are definable in terms of the corresponding **MQ**- (resp. **HQ**-) primitives $\sqcup$, $\amalg$, resp., and the **HQ**-operator ω. In $\lambda\gamma_{\&}$**CQ** this situation can be expressed trivially by specializing $[\beta\bigvee_{\natural}\bot]$, $[\beta\bigvee_{\cup}\bot]$, resp., *modulo* β-conversion $[\beta \to \lambda]$.

Lemma (**HQ**-*admissible instances of the* $[\bigvee_{\natural}$-$,\bigvee_{\cup}]$-*selectors*).

$[\beta\sqcup\bot]$ $\Gamma \vdash \omega_{\mathrm{C}}(\sqcup(\mathrm{h},[\mathrm{x{:}A}].\mathrm{e}_1[\![\mathrm{x}]\!]{:}\bot,[\mathrm{y{:}B}].\mathrm{e}_2[\![\mathrm{y}]\!]{:}\bot) = \bigvee_{\natural}(\mathrm{z}{:}\neg\mathrm{C}).\mathrm{h} \diamondsuit [\lambda\mathrm{x{:}A}.\mathrm{e}_1[\![\mathrm{x}]\!], \lambda\mathrm{y{:}B}.\mathrm{e}_2[\![\mathrm{y}]\!]]$ $[:\ \mathrm{C}]$,
if $\Gamma \vdash \mathrm{h} : \mathrm{A} \vee \mathrm{B}$, $\Gamma[\mathrm{x{:}A}] \vdash \mathrm{e}_1[\![\mathrm{x}]\!] : \bot$, $\Gamma[\mathrm{y{:}B}] \vdash \mathrm{e}_2[\![\mathrm{y}]\!] : \bot$,
provided $\mathrm{z} \notin \mathrm{FV}_{\lambda}(\mathrm{e}_1[\![\mathrm{x}]\!],\mathrm{e}_2[\![\mathrm{y}]\!])$, $[\mathrm{x},\ \mathrm{y},\ \mathrm{z} \notin \mathrm{FV}_{\lambda}(\mathrm{h})]$,

$[\beta\amalg\bot]$ $\Gamma \vdash \omega_{\mathrm{C}}(\amalg(\mathrm{h},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}[\![\mathrm{u}]\!]].\mathrm{e}[\![\mathrm{u,x}]\!]{:}\bot)) = \bigvee_{\cup}(\mathrm{z}{:}\neg\mathrm{C}).\mathrm{h} \diamondsuit [!\mathrm{u}.\lambda\mathrm{x{:}A}.\mathrm{e}[\![\mathrm{u,x}]\!]]$ $[:\ \mathrm{C}]$,
if $\Gamma \vdash \mathrm{h} : \exists\mathrm{u}.\mathrm{A}[\![\mathrm{u}]\!]$, $\Gamma[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}] \vdash \mathrm{e}[\![\mathrm{u,x}]\!] : \bot$,
provided $\mathrm{z} \notin \mathrm{FV}_{\lambda}(\mathrm{e}[\![\mathrm{u,x}]\!])$, $[\mathrm{x},\ \mathrm{y},\ \mathrm{z} \notin \mathrm{FV}_{\lambda}(\mathrm{h}), \text{etc.}]$, $[\mathrm{u} \notin \mathrm{FV}_{u}(\mathrm{C})]$,

Proof. $[\beta\sqcup\bot]$: Use $[\beta \to \lambda]$, and $[\beta\bigvee_{\natural}\bot]$. $[\beta\amalg\bot]$: Analogously, by $[\beta \to \lambda]$, and $[\beta\bigvee_{\cup}\bot]$. □

We can go now into the proper **HQ**-"commuting" rules. As expected, these must be special cases of conversions described in $\lambda\gamma_{\&}$**CQ**, as "$[\bigvee_{\natural},\bigvee_{\cup}]$-diagonal" situations. We can show, in fact, that $\lambda\gamma_{\&}$**CQ** *is equationally complete with respect to the Heyting "commuting conversions"*. This is as it should be.

(1) The simplest Heyting $\vee$- and $\exists$-"commuting" conversions are those relative to ω-terms. [Recall that the ω-operators count as selectors in **HQ**.] These are instances of appropriate derived $\lambda\gamma_{\&}$**CQ**-rules.

Corollary *(The Heyting* $[\vee\omega,\exists\omega]$*-commuting conversions in* $\lambda\gamma_{\&}$**CQ**).

(1) $[\beta\vee\omega]_h \equiv [\beta\bigvee_{\natural}\bot\text{-}\oint\bigvee_{\natural}\omega\omega]_h \equiv [\beta\vee\bot]_h$:

$[\beta\vee\omega]_h$ $\Gamma \vdash \omega_{\mathrm{C}}(\sqcup(\mathrm{h},[\mathrm{x{:}A}].\mathrm{a}[\![\mathrm{x}]\!]{:}\bot,[\mathrm{y{:}B}].\mathrm{b}[\![\mathrm{y}]\!]{:}\bot)) = \sqcup(\mathrm{h},[\mathrm{x{:}A}].\omega_{\mathrm{C}}(\mathrm{a}[\![\mathrm{x}]\!]){:}\mathrm{C},[\mathrm{y{:}B}].\omega_{\mathrm{C}}(\mathrm{b}[\![\mathrm{y}]\!]){:}\mathrm{C}) : \mathrm{C}$,

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[23] Were we not interested in isolating the general Boolean equational proof-patterns of the Heyting (conversion-) rules first, the latter could have been available by applying blindly the definitions and by computing resulting proof-term expansions, a good job for a machine, say. [Actually, a $\lambda\gamma$-system can be easily implemented as a λ-calculus "reduction machine", endowed with a relatively simple type-checking facility.] Of course, the main point of the present notes is that of showing that there is some *structure* in the game; viz. that the genuine proof-conversions of the Heyting calculus are, indeed, *special cases* – and oft just *instances* – of general "proof-isomorphisms" (matching, e.g., various β-"evaluation" patters, extensionality principles, "diagonal situations", etc.) that obtain in [*extensional*] $\lambda\gamma$-calculi. As we show next, this is – technically – unproblematic. In the epistemic order of things, the difficulty appears in the attempt to *qualify the specifics* of the Heyting conversion-rules, within the classical proof-realm, or yet, more generally, while looking for a would-be *criterion of construction* behind the Heyting calculus. The outcome of this section would, apparently, leave the things in disorder, so that the Heyting proof-calculus looks, *prima facie*, very composite on the equational level. [Some authors have even based their attack against "intuitionistic logic" on this first impression, pointing out to things like, "the *ad hoc* character of the rules".] The fact is that *exactly* this choice of an equational proof-system is *forced upon*, by Brouwer's *genuine* views on *difference* and *negation*, views that are *counter*-traditional, so to speak. The attempt to settle down this type of problem makes appeal to a different conceptual archaeology, however, and is discussed elsewhere.

$\quad\quad$ if $\Gamma \vdash \mathrm{h} : \mathrm{A} \vee \mathrm{B}, \Gamma[\mathrm{x{:}A}] \vdash \mathrm{a}\llbracket\mathrm{x}\rrbracket : \bot, \Gamma[\mathrm{y{:}B}] \vdash \mathrm{b}\llbracket\mathrm{y}\rrbracket : \bot,$

(2) $[\beta\exists\omega]_h \equiv [\beta\bigvee_{\cup}\bot\text{-}\oint\bigvee_{\cup}\omega]_h \equiv [\beta\exists\bot]_h$:

$[\beta\exists\omega]_h \quad \Gamma \vdash \omega_{\mathrm{C}}(\amalg(\mathrm{h},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}].\mathrm{c}\llbracket\mathrm{u,x}\rrbracket{:}\bot)) = \amalg(\mathrm{h},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}].\omega_{\mathrm{C}}(\mathrm{c}\llbracket\mathrm{u,x}\rrbracket){:}\mathrm{C}) : \mathrm{C},$
if $\Gamma \vdash \mathrm{h} : \exists\mathrm{u.A}\llbracket\mathrm{u}\rrbracket, \Gamma[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket] \vdash \mathrm{c}\llbracket\mathrm{u,x}\rrbracket : \bot.$

Proof. These are special cases of $[\beta\vee\gamma] \equiv [\beta\bigvee_{\natural}\bot\text{-}\oint\bigvee_{\natural}\gamma\gamma]$ and $[\beta\exists\gamma] \equiv [\beta\bigvee_{\cup}\bot\text{-}\oint\bigvee_{\cup}\gamma]$. (The latter have been obtained as $\lambda\gamma_{\&}\mathbf{CQ}$-consequences of $[\oint\bigvee_{\natural}\gamma\gamma]$ and $[\oint\bigvee_{\cup}\gamma]$, resp., *modulo* $[\beta\bigvee_{\natural}\bot]$ and $[\beta\bigvee_{\cup}\bot]$.) □

(2) *Modulo* applications of $[\beta \to \lambda]$-conversion (i.e., the usual β-rule of the ordinary typed λ-calculus), the following are instances of the positive $[\beta\bigvee_{\natural}]$- resp. $[\beta\bigvee_{\cup}]$-rules of $\lambda\gamma_{\&}\mathbf{CQ}$.

Corollary (*Positive* $\sqcup(\vee)$-*commuting conversions in* $\lambda\gamma_{\&}\mathbf{CQ}$).

(1) $[\beta\sqcup\!\to]_h \equiv [\beta\vee\!\to]_h$:

$\Gamma \vdash \sqcup(\mathrm{h},[\mathrm{x{:}A}].\mathrm{e}_1\llbracket\mathrm{x}\rrbracket{:}(\mathrm{F}{\to}\mathrm{G}),[\mathrm{y{:}B}].\mathrm{e}_2\llbracket\mathrm{y}\rrbracket{:}(\mathrm{F}{\to}\mathrm{G}))(\mathrm{f}) = \sqcup(\mathrm{h},[\mathrm{x{:}A}].\mathrm{e}_1\llbracket\mathrm{x}\rrbracket(\mathrm{f}){:}\mathrm{G},[\mathrm{y{:}B}].\mathrm{e}_2\llbracket\mathrm{y}\rrbracket(\mathrm{f}){:}\mathrm{G})\ [{:}\ \mathrm{G}],$
if $\Gamma \vdash \mathrm{h} : \mathrm{A} \vee \mathrm{B}, \Gamma \vdash \mathrm{f} : \mathrm{F}, \Gamma[\mathrm{x{:}A}] \vdash \mathrm{e}_1\llbracket\mathrm{x}\rrbracket : \mathrm{F} \to \mathrm{G}, \Gamma[\mathrm{y{:}B}] \vdash \mathrm{e}_2\llbracket\mathrm{y}\rrbracket : \mathrm{F} \to \mathrm{G},$

(2) $[\beta\sqcup\wedge_1]_h \equiv [\beta\vee\wedge_1]_h$:

$\Gamma \vdash \mathbf{p}_1(\sqcup(\mathrm{h},[\mathrm{x{:}A}].\mathrm{e}_1\llbracket\mathrm{x}\rrbracket{:}(\mathrm{F}{\wedge}\mathrm{G}),[\mathrm{y{:}B}].\mathrm{e}_2\llbracket\mathrm{y}\rrbracket{:}(\mathrm{F}{\wedge}\mathrm{G}))) = \sqcup(\mathrm{h},[\mathrm{x{:}A}].\mathbf{p}_1(\mathrm{e}_1\llbracket\mathrm{x}\rrbracket){:}\mathrm{F},[\mathrm{y{:}B}].\mathbf{p}_1(\mathrm{e}_2\llbracket\mathrm{y}\rrbracket){:}\mathrm{F})\ [{:}\ \mathrm{F}],$
if $\Gamma \vdash \mathrm{h} : \mathrm{A} \vee \mathrm{B}, \Gamma[\mathrm{x{:}A}] \vdash \mathrm{e}_1\llbracket\mathrm{x}\rrbracket : \mathrm{F} \wedge \mathrm{G}, \Gamma[\mathrm{y{:}B}] \vdash \mathrm{e}_2\llbracket\mathrm{y}\rrbracket : \mathrm{F} \wedge \mathrm{G},$

(3) $[\beta\sqcup\wedge_2]_h \equiv [\beta\vee\wedge_2]_h$:

$\Gamma \vdash \mathbf{p}_2(\sqcup(\mathrm{h},[\mathrm{x{:}A}].\mathrm{e}_1\llbracket\mathrm{x}\rrbracket{:}(\mathrm{F}{\wedge}\mathrm{G}),[\mathrm{y{:}B}].\mathrm{e}_2\llbracket\mathrm{y}\rrbracket{:}(\mathrm{F}{\wedge}\mathrm{G}))) = \sqcup(\mathrm{h},[\mathrm{x{:}A}].\mathbf{p}_2(\mathrm{e}_1\llbracket\mathrm{x}\rrbracket){:}\mathrm{G},[\mathrm{y{:}B}].\mathbf{p}_2(\mathrm{e}_2\llbracket\mathrm{y}\rrbracket){:}\mathrm{G})\ [{:}\ \mathrm{G}],$
if $\Gamma \vdash \mathrm{h} : \mathrm{A} \vee \mathrm{B}, \Gamma[\mathrm{x{:}A}] \vdash \mathrm{e}_1\llbracket\mathrm{x}\rrbracket : \mathrm{F} \wedge \mathrm{G}, \Gamma[\mathrm{y{:}B}] \vdash \mathrm{e}_2\llbracket\mathrm{y}\rrbracket : \mathrm{F} \wedge \mathrm{G},$

(4) $[\beta\sqcup\forall]_h \equiv [\beta\vee\forall]_h$: **where t** is free for u in [F ≡] $\mathrm{F}\llbracket\mathrm{u}\rrbracket$, and $\mathrm{F}\llbracket\mathbf{t}\rrbracket \equiv \mathrm{F}\llbracket\mathrm{u}{:=}\mathbf{t}\rrbracket$,

$\Gamma \vdash (\sqcup(\mathrm{h},[\mathrm{x{:}A}].\mathrm{e}_1\llbracket\mathrm{x}\rrbracket{:}\forall\mathrm{u.F},[\mathrm{y{:}B}].\mathrm{e}_2\llbracket\mathrm{y}\rrbracket{:}\forall\mathrm{u.F}))[\mathbf{t}] = \sqcup(\mathrm{h},[\mathrm{x{:}A}].\mathrm{e}_1\llbracket\mathrm{x}\rrbracket[\mathbf{t}]{:}\mathrm{F}\llbracket\mathbf{t}\rrbracket,[\mathrm{y{:}B}].\mathrm{e}_2\llbracket\mathrm{x}\rrbracket[\mathbf{t}]{:}\mathrm{F}\llbracket\mathbf{t}\rrbracket)\ [{:}\ \mathrm{F}\llbracket\mathbf{t}\rrbracket],$
if $\Gamma \vdash \mathrm{h} : \mathrm{A} \vee \mathrm{B}, \Gamma \Vdash \mathbf{t} :: \mathbf{U}, \Gamma[\mathrm{x{:}A}] \vdash \mathrm{e}_1\llbracket\mathrm{x}\rrbracket : \forall\mathrm{u.F}\llbracket\mathrm{u}\rrbracket, \Gamma[\mathrm{y{:}B}] \vdash \mathrm{e}_2\llbracket\mathrm{y}\rrbracket : \forall\mathrm{u.F}\llbracket\mathrm{u}\rrbracket.$

Proof. [Note that, in each case, x, y $\notin \mathrm{FV}_\lambda(\mathrm{h})$.] $[\beta\sqcup\!\to]_h$: *Modulo* $[\beta \to \lambda]$-conversion, this is a special case of $[\beta\bigvee_{\natural}\!\to]$. $[\beta\sqcup\wedge_1]_h$, $[\beta\sqcup\wedge_2]_h$: Analogously, using $[\beta\bigvee_{\natural}\wedge_1]$ resp. $[\beta\bigvee_{\natural}\wedge_2]$ and $[\beta \to \lambda]$). $[\beta\sqcup\forall]_h$: By $[\beta\bigvee_{\natural}\forall]$ and $[\beta \to \lambda]$. The details can be safely left as *exercises* to the reader. □

Corollary (*Positive* $\amalg(\exists)$-*commuting conversions in* $\lambda\gamma_{\&}\mathbf{CQ}$).

(1) $[\beta\amalg\!\to]_h \equiv [\beta\exists\!\to]_h$:

$\Gamma \vdash (\amalg(\mathrm{h} \diamondsuit [\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket].\mathrm{e}\llbracket\mathrm{u,x}\rrbracket{:}(\mathrm{F}{\to}\mathrm{G}))(\mathrm{f}) = \amalg(\mathrm{h},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket].\mathrm{e}\llbracket\mathrm{u,x}\rrbracket(\mathrm{f}){:}\mathrm{G})\ [{:}\ \mathrm{G}],$
if $\Gamma \vdash \mathrm{f} : \mathrm{F}, \Gamma \vdash \mathrm{h} : \exists\mathrm{u.A}\llbracket\mathrm{u}\rrbracket, \Gamma[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket] \vdash \mathrm{e}\llbracket\mathrm{u,x}\rrbracket : \mathrm{F} \to \mathrm{G}$, *provided* $\mathrm{u} \notin \mathrm{FV}_u(\mathrm{F,G})$,

(2) $[\beta\amalg\wedge_1]_h \equiv [\beta\exists\wedge_1]_h$:

$\Gamma \vdash \mathbf{p}_1(\amalg(\mathrm{h},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket].\mathrm{e}\llbracket\mathrm{u,x}\rrbracket{:}(\mathrm{F}{\wedge}\mathrm{G}))) = \amalg(\mathrm{h},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket].\mathbf{p}_1(\mathrm{e}\llbracket\mathrm{u,x}\rrbracket){:}\mathrm{F})\ [{:}\ \mathrm{F}],$
if $\Gamma \vdash \mathrm{h} : \exists\mathrm{u.A}\llbracket\mathrm{u}\rrbracket, \Gamma[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket] \vdash \mathrm{e}\llbracket\mathrm{u,x}\rrbracket : \mathrm{F} \wedge \mathrm{G}$, *provided* $\mathrm{u} \notin \mathrm{FV}_u(\mathrm{F,G})$,

(3) $[\beta\amalg\wedge_2]_h \equiv [\beta\exists\wedge_2]_h$:

$\Gamma \vdash \mathbf{p}_2(\amalg(\mathrm{h},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket].\mathrm{e}\llbracket\mathrm{u,x}\rrbracket{:}(\mathrm{F}{\wedge}\mathrm{G}))) = \amalg(\mathrm{h},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket].\mathbf{p}_2(\mathrm{e}\llbracket\mathrm{u,x}\rrbracket){:}\mathrm{G})\ [{:}\ \mathrm{G}],$
if $\Gamma \vdash \mathrm{h} : \exists\mathrm{u.A}\llbracket\mathrm{u}\rrbracket, \Gamma[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket] \vdash \mathrm{e}\llbracket\mathrm{u,x}\rrbracket : \mathrm{F} \wedge \mathrm{G}$, *provided* $\mathrm{u} \notin \mathrm{FV}_u(\mathrm{F,G})$,

(4) $[\beta\amalg\forall]_h \equiv [\beta\exists\forall]_h$: **where t** is free for v in $\mathrm{F}\llbracket\mathrm{v}\rrbracket$, and $\mathrm{F}\llbracket\mathbf{t}\rrbracket \equiv \mathrm{F}\llbracket\mathrm{v}{:=}\mathbf{t}\rrbracket$,

$\Gamma \vdash \amalg(\mathrm{h},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket].\mathrm{e}\llbracket\mathrm{u,x}\rrbracket{:}(\forall\mathrm{v.F}\llbracket\mathrm{v}\rrbracket))[\mathbf{t}] = \amalg(\mathrm{h},[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket].\mathrm{e}\llbracket\mathrm{u,x}\rrbracket[\mathbf{t}]{:}\mathrm{F}\llbracket\mathbf{t}\rrbracket)\ [{:}\ \mathrm{F}\llbracket\mathbf{t}\rrbracket]$
if $\Gamma \Vdash \mathbf{t} :: \mathbf{U}, \Gamma \vdash \mathrm{h} : \exists\mathrm{u.A}\llbracket\mathrm{u}\rrbracket, \Gamma[\mathrm{u}{:}\mathbf{U}][\mathrm{x{:}A}\llbracket\mathrm{u}\rrbracket] \vdash \mathrm{e}\llbracket\mathrm{u,x}\rrbracket : \forall\mathrm{v.F}\llbracket\mathrm{v}\rrbracket$, *provided* $\mathrm{u} \notin \mathrm{FV}_u(\forall\mathrm{v.F}\llbracket\mathrm{v}\rrbracket)$.

Proof. [One has, in each case, x $\notin \mathrm{FV}_\lambda(\mathrm{h})$, while, in (1), x $\notin \mathrm{FV}_\lambda(\mathrm{f})$], too.] Use $[\beta\bigvee_{\cup}\!\to]$, $[\beta\bigvee_{\cup}\wedge_1]$, $[\beta\bigvee_{\cup}\wedge_2]$, $[\beta\bigvee_{\cup}\forall]$ resp., and $[\beta \to \lambda]$. □

(3) Finally, the following rules are instances of the appropriate $[\oint\bigvee_{\natural}\bigvee_{\natural}]$-, $[\oint\bigvee_{\cup}\bigvee_{\natural}]$-, $[\oint\bigvee_{\natural}\bigvee_{\cup}]$- and $[\oint\bigvee_{\cup}\bigvee_{\cup}]$-"diagonal" rules of $\lambda\gamma_{\&}\mathbf{CQ}$.

Corollary (*Negative* $\sqcup(\vee)$*-commuting conversions in* $\lambda\gamma_{\&}\mathbf{CQ}$).

(1) $[\oint\sqcup\sqcup{:}\vee\vee]_h \equiv [\beta\vee\vee]_h$:

$\Gamma \vdash \sqcup(c \diamondsuit [x_0{:}F].e_1[\![x_0]\!]{:}E,[y_0{:}G].e_2[\![y_0]\!]{:}E) = \sqcup(h \diamondsuit [x{:}A].c_1[\![x]\!]{:}E,[y{:}B].c_2[\![y]\!]{:}E)$ [: E]
if $\Gamma \vdash h : A \vee B$, $\Gamma[x{:}A] \vdash a[\![x]\!] : F \vee G$, $\Gamma[y{:}B] \vdash b[\![y]\!] : F \vee G$,
and $\Gamma[x_0{:}F] \vdash e_1[\![x_0]\!] : E$, $\Gamma[y_0{:}G] \vdash e_2[\![y_0]\!] : E$,

where $[x, y \notin FV_\lambda(h, e_1[\![x]\!],e_2[\![y]\!])]$, $[x_0, y_0 \notin FV_\lambda(h,a[\![x]\!],b[\![y]\!])]$,
$c \equiv \sqcup(h,[x{:}A].a[\![x]\!]{:}(F\vee G),[y{:}B].b[\![y]\!]{:}(F \vee G))$,
$c_1[\![x]\!] \equiv \sqcup(a[\![x]\!],[x_0{:}F].e_1[\![x_0]\!]{:}E,[y_0{:}G].e_2[\![y_0]\!]{:}E)$, and $c_2[\![y]\!] \equiv \sqcup(b[\![y]\!],[x_0{:}F].e_1[\![x_0]\!]{:}E,[y_0{:}G].e_2[\![y_0]\!]{:}E)$,

(2) $[\oint\amalg\sqcup{:}\exists\exists]_h \equiv [\beta\vee\exists]_h$:

$\Gamma \vdash \amalg(c,[u{:}\mathbf{U}][x_0{:}F].e[\![u,x_0]\!]{:}E) = \sqcup(h,[x{:}A].e_1[\![x]\!]{:}E,[y{:}B].e_2[\![y]\!]{:}E)$ [: E],
if $\Gamma \vdash h : A \vee B$, $\Gamma[x{:}A] \vdash a[\![x]\!] : \exists u.F[\![u]\!]$, $\Gamma[y{:}B] \vdash b[\![y]\!] : \exists u.F[\![u]\!]$, $\Gamma[u{:}\mathbf{U}][x_0{:}F[\![u]\!]] \vdash e[\![u,x_0]\!] : E$,

where $[x_0 \notin FV_\lambda(h,a[\![x]\!],b[\![y]\!])]$, $[x, y \notin FV_\lambda(h,e[\![u,x_0]\!])]$, provided $u \notin FV_u(E)$,
$c \equiv \sqcup(h,[x{:}A].a[\![x]\!]{:}(\exists u.F[\![u]\!]),[y{:}B].b[\![y]\!]{:}(\exists u.F[\![u]\!]))$,
$e_1[\![x]\!] \equiv \amalg(a[\![x]\!],[u{:}\mathbf{U}][x_0{:}F[\![u]\!]].e[\![u,x_0]\!]{:}E)$, and $e_2[\![y]\!] \equiv \amalg(b[\![y]\!],[u{:}\mathbf{U}][x_0{:}F[\![u]\!]].e[\![u,x_0]\!]{:}E)$.

Proof. $[\oint\sqcup\sqcup{:}\vee\vee]_h$: This is an instance of $[\oint\bigvee_\natural\bigvee_\natural{:}\vee\vee]$. Here, ($[\beta \to \lambda]$ must be used in order to eliminate p-terms of the form $\Omega(d) \equiv (\lambda z{:}\bot.z)(d)$. $[\oint\amalg\sqcup{:}\exists\exists]_h$: Analogously [*exercise*], using $[\oint\bigvee_\cup\bigvee_\natural{:}\exists\exists]$ and $[\beta \to \lambda]$. □

Corollary (*Negative* $\amalg(\exists)$*-commuting conversions in* $\lambda\gamma_{\&}\mathbf{CQ}$).

(1) $[\oint\sqcup\amalg{:}\vee]_h \equiv [\beta\exists\vee]_h$:

$\Gamma \vdash \sqcup(c,[x_0{:}F].e_1[\![x_0]\!]{:}E,[y_0{:}G].e_2[\![y_0]\!]{:}E) = \amalg(h,[u{:}\mathbf{U}][x{:}A[\![u]\!]].e[\![u,x]\!]{:}E)$ [: E],
if $\Gamma \vdash h : \exists u.A[\![u]\!]$, $\Gamma[u{:}\mathbf{U}][x{:}A[\![u]\!]] \vdash a[\![u,x]\!] : F \vee G$, $\Gamma[x_0{:}F] \vdash e_1[\![x_0]\!] : E$, $\Gamma[y_0{:}G] \vdash e_2[\![y_0]\!] : E$,

where $[x_0, z_0 \notin FV_\lambda(h,a[\![u,x]\!])]$, $[x \notin FV_\lambda(h,e_1[\![x_0]\!],e_2[\![y_0]\!])]$, $[u \notin FV_u(E,e_1[\![x_0]\!],e_2[\![y_0]\!])]$,
$c \equiv \amalg(h,[u{:}\mathbf{U}][x{:}A[\![u]\!]].a[\![u,x]\!]{:}(F\vee G))$, and $e[\![u,x]\!] \equiv \sqcup(a[\![u,x]\!],[x_0{:}F].e_1[\![x_0]\!]{:}E,[y_0{:}G].e_2[\![y_0]\!]{:}E)$,
provided $u \notin FV_u(F,G)$,

(2) $[\oint\amalg\amalg{:}\exists]_h \equiv [\beta\exists\exists]_h$:

$\Gamma \vdash \amalg(c,[v{:}\mathbf{U}][x_0{:}F[\![v]\!]].e[\![v,x_0]\!]{:}E) = \amalg(h,[u{:}\mathbf{U}][x{:}A[\![u]\!]].b[\![u,x]\!]{:}E)$ [: E],
if $\Gamma \vdash h : \exists u.A[\![u]\!]$, $\Gamma[u{:}\mathbf{U}][x{:}A[\![u]\!]] \vdash a[\![u,x]\!] : \exists v.F[\![v]\!]$, $\Gamma[v{:}\mathbf{U}][x_0{:}F[\![v]\!]] \vdash e[\![v,x_0]\!] : E$,

where $[x_0 \notin FV_\lambda(h,a[\![u,x]\!])]$, $[x \notin FV_\lambda(h,e[\![u,x_0]\!])]$, $[u \notin FV_u(e[\![v,x_0]\!])]$, $[v \notin FV_u(a[\![u,x]\!])]$,
$c \equiv \amalg(h,[u{:}\mathbf{U}][x{:}A[\![u]\!]].a[\![u,x]\!]{:}(\exists v.F[\![v]\!]))$, $b[\![u,x]\!] \equiv \amalg(a[\![u,x]\!],[v{:}\mathbf{U}][x_0{:}F[\![v]\!]].e[\![v,x_0]\!]{:}E)$,
provided $u \notin FV_u(\exists v.F[\![v]\!])$ and $v \notin FV_u(E)$.

Proof. Equally easy [*exercise*] (for $[\oint\sqcup\amalg{:}\vee]_h$: use $[\oint\bigvee_\natural\bigvee_\cup{:}\vee]$ and $[\beta \to \lambda]$; for $[\oint\amalg\amalg{:}\exists]_h$: use $[\oint\bigvee_\cup\bigvee_\cup{:}\exists]$, $[\beta \to \lambda]$; the β-rule eliminates p-terms $f[\![u,x]\!](d)$, with $\Gamma[u{:}\mathbf{U}][x{:}A[\![u]\!]] \vdash f[\![u,x]\!] \equiv_{df} \Omega : \top$). □

Remark (*Intensional variants of the* $[\vee,\exists]$*-commuting* **HQ**-*rules*). It is relatively easy (although rather tedious) to establish *effectively* the fact that "intensional" analogues of the Heyting $[\vee,\exists]$-commuting rules are already available in $\lambda\gamma\mathbf{CQ}$ (the $\wedge$-free subsystem of $\lambda\gamma_{\&}\mathbf{CQ}$, defined on $\vdash[\mathbf{CQ}]$, although *not* in its diagonalization-free fragment $\lambda\gamma!$) in terms of Boolean intensional algebraic proof-operators associated to any one of the $[\otimes,\oplus]$-pairs defined previously [*exercise*]. (Globally, this follows also from the remark that $[\eta\wedge]$ and $[\eta\forall]$ were not needed in the derivation of the more general Boolean analogues.) Notably, if we leave out $[\oint\gamma]$ (i.e., if we work in the diagonalization-free subsystem $\lambda\gamma!$ of $\lambda\gamma\mathbf{CQ}$), only "cross-diagonal situations" are lost. For the **HQ**-case, of concern here, this means that we loose just the *negative* $[\sqcup(\vee),\amalg(\exists)]$*-conversions*, viz. $[\beta\vee\vee]_h$, $[\beta\vee\exists]_h$, $[\beta\exists\vee]_h$, $[\beta\exists\exists]_h$.

The Heyting proof-calculus $\lambda\mathbf{HQ}$. We can finally isolate the proof-theory of Heyting's first-order logic **HQ** as a proper sub-theory of $\lambda\gamma_{\&}\mathbf{CQ}$. As is well-known, at a provability-level, one has $\mathbf{HQ} \subset \mathbf{CQ}$ (proper inclusion for the corresponding sets of provable formulas). In fact, as in the case of Johansson's *Minimalkalkül*, *the Heyting logic* **HQ** can be viewed as *a proper fragment of* **CQ**, at a *proof-theoretic* level, *modulo* an appropriate *definitional embedding*.

Relative to the [⊥,→,∧,∨,∀,∃] type-structure, the *proof-syntax* of the first-order Heyting logic **HQ**, the *stratification* ⊢[**HQ**] of the *Heyting proof-terms*, and the *equational system* of the Heyting *first-order proof-calculus* can be described along a pattern matching, in the obvious way, the previous criteria used to display the Boolean and "minimal" proof-syntax. Summing up, the proof-theory of Heyting's first-order logic **HQ** [*"the Heyting proof-calculus"*] consists of the following cocktail:

Definition (⊢[**HQ**] *and* λ**HQ**).

(1) *Heyting proof-operators*: as for ⊢[**MQ**], with, moreover,
- *inferential operators*: $\omega_A(\dots)$, i.e., a family (of *sumptors*) indexed on A in [⊥,→,∧,∨,∀,∃],

(2) *Heyting proof-terms* [primitive forms], as for ⊢[**MQ**], with, moreover:
- *inferential* forms: $\omega_A(e{:}\bot)$ [$\equiv \omega(e{:}\bot) : A$],

(3) *Heyting proof-rules* (*the* ⊢[**HQ**]*-stratification*):
- *proof-context rules*: as for $\vdash_*$[**CQ**] and/or ⊢[**MQ**],
- *"type-assignment" rules*: as for ⊢[**MQ**], with an additional *inferential rule*:
  $(\to i\omega)_h$ $\Gamma \vdash e : \bot \Rightarrow \Gamma \vdash \omega_A(e) : A$.

(4) λ**HQ** is an equational theory defined on ⊢[**HQ**] by the "*Heyting proof-conversion rules*":
- λ**MQ**-*conversions* [=*Minimalkalkül*]: as for ⊢[**MQ**],
  - *inferential* (*Boolean*): [$\beta\to\lambda$], [$\eta\to\lambda$], and [$\xi\to\lambda$], [$\mu\to$], [$\nu\to$],
  - *algebraic*:
    Boolean: [$\beta\wedge_1$], [$\beta\wedge_2$], [$\eta\wedge$], and [$\xi\wedge$], [$\nu\wedge_1$], [$\nu\wedge_2$],
    "minimal": $[\beta\vee_1]_m$, $[\beta\vee_2]_m$, $[\eta\vee]_m$, and $[\xi\vee_1]_m$, $[\xi\vee_2]_m$, $[\mu\vee]_m$,
  - *generic*:
    Boolean: [$\beta\forall$], [$\eta\forall$], and [$\xi\forall$], [$\mu\forall$],
    "minimal": $[\beta\exists]_m$, $[\eta\exists]_m$, and $[\xi\exists]_m$, $[\mu\exists]_m$,
- ω-*rules* (*"ω-conversions"*) [specific, beyond λ**MQ**]:
  - $[\beta\omega]_h$ ($= [\beta\omega\bot]_h$, [$\beta\omega\to$], [$\beta\omega\wedge_1$], [$\beta\omega\wedge_2$], $[\beta\omega\vee]_h$, [$\beta\omega\forall$], $[\beta\omega\exists]_h$), i.e.,
    Boolean βω-conversions: [$\beta\omega\to$], [$\beta\omega\wedge_1$], [$\beta\omega\wedge_2$], [$\beta\omega\forall$],
    Brouwerian βω-conversions: $[\beta\omega\bot]_h$, $[\beta\omega\vee]_h$, $[\beta\omega\exists]_h$,
  - $[\eta\omega]_h$ ($\equiv [\eta\bot\omega]_h \equiv [\beta\omega\bot]$, "⊥-extensionality", *Boolean*),
  - [$\nu\omega$] = ("ω-congruence"),
- *"Heyting commuting conversions"* [specific, beyond λ**MQ**]:
  - *Brouwerian*, [$\omega(\bot)$]-*conversions*: $[\beta\vee\bot]_h \equiv [\beta\vee\omega]_h$, $[\beta\exists\bot]_h \equiv [\beta\exists\omega]_h$,
  - *positive* ⊔(∨)-*conversions* (i := 1,2): $[\beta\vee\to]_h \equiv [\beta\sqcup\to]$, $[\beta\vee\wedge_i]_h \equiv [\beta\sqcup\wedge_i]$, $[\beta\vee\forall]_h \equiv [\beta\sqcup\forall]$,
  - *positive* II(∃)-*conversions* (i := 1,2): $[\beta\exists\to]_h \equiv [\beta\text{II}\to]$, $[\beta\exists\wedge_i]_h \equiv [\beta\text{II}\wedge_i]$, $[\beta\exists\forall]_h \equiv [\beta\text{II}\forall]$,
  - *negative* ⊔(∨)-*conversions*: $[\beta\vee\vee]_h \equiv [\oint\sqcup\sqcup{:}\vee\vee]$, $[\beta\vee\exists]_h \equiv [\oint\text{II}\sqcup{:}\exists\exists]$,
  - *negative* II(∃)-*conversions*: $[\beta\exists\vee]_h \equiv [\oint\sqcup\text{II}{:}\vee]$, $[\beta\exists\exists]_h \equiv [\oint\text{IIII}{:}\exists]$.

So, the "positive" stratification ⊢[**MQ**] can be obtained from ⊢[**HQ**], by leaving out the ω-primitive(s) and the associated inferential rule $(\to i\omega)_h$. Analogously, λ**MQ** is, equationally, like λ**HQ**, except for the fact that it lacks the ω-rules ("ω-conversions") and the "commuting conversions" (in the present setting, $21 = 7 \times 3$ "postulates").

Remark (*The proof-conversion rules of* λ**HQ**).

(1) Since, ignoring congruence conditions, the theory λ**MQ** is given by a dozen of equations as "postulates" (exactly), the (formal) proof-theory of first-order intuitionistic logic (λ**HQ**) requires 33 [viz., 12 + 21]

"postulates". For the record, $\lambda\pi!$ has only 7 ($\beta\eta$-) "postulates", whereas the proof-theory of first-order classical logic – in its most economic form, i.e., $\lambda\gamma_0$**CQ** – needs an extra pair $[\oint \gamma]_0$, $[\eta \to \gamma]_0$ $(7+2=9)$. As expected, intuitionistic proof-theory is quite *expensive*.

(2) *Genuinely "Brouwerian" conversion-rules*. The classification of the λ**HQ**-rules above makes also clear the fact that the "purely" Brouwerian ideas are not too many and play a minor rôle in the economy of the full game. They concern the special cases of the γ-applications (uses of *reductio ad absurdum*), viz., those that are also "reliable" [Dutch: *betrouwbaar*] for the intuitionist. Specifically, these are the "$\beta\omega$-*conversions*" in $[\beta\omega]_h$ (of which only $[\beta\omega\bot]_h$, $[\beta\omega\vee]_h$, $[\beta\omega\exists]_h$ are genuinely "*Brouwerian*") and the (*Brouwerian*) $[\omega(\bot)]$-*conversions*: $[\beta\vee\bot]_h$ and $[\beta\exists\bot]_h$. (The "$\bot$-extensionality" property $[\eta\omega]_h \equiv [\eta\bot\omega]_h \equiv [\beta\omega\bot]$ is, after all, a *Boolean* rule.)

(3) *Historically*, the specific "Heyting" rules (the $_h$-rules) that are not already "Brouwerian" should be rather credited to Dag Prawitz [65]. Of course, on a pure provability-level, Johansson [36] is also indebted to Heyting [30], and, as we can learn from Heyting himself (see, e.g., [Troelstra 78], [van Stigt 90] and the relevant Heyting correspondence with Oskar Becker quoted in [Troelstra 81]), Heyting is ultimately indebted – somewhat empirically – to *Principia Mathematica*! Otherwise, the "positive"/"minimal" logic has been also anticipated by Jan Łukasiewicz, David Hilbert and Paul Hertz (1881-1940), one of the collaborators of Hilbert on physics in Göttingen (1912-1933). But such rudiments were not (yet) "proof-theory", in the sense of the present notes. The "**BHK**"-reading of the **HQ**-proofs is, in fact, the result of a *systematic reflection on the meaning of proving*, with the *proofs* as occurring in the mathematical practice of the intuitionist, *rather than* the consequence of a formal study of *provability* [a property that applies to propositions and/or formulas, not to proofs and/or proof-terms] in **HQ**.

Since the most general Boolean *positive selectors* $\bigwedge_\vdash$, $\bigwedge_\natural$, and $\bigwedge_\cup$ make also sense intuitionistically, one can define them in $\vdash$[**HQ**] like in the classical case (and/or in $\vdash$[**MQ**]).

Of course, $\vdash$[**HQ**] can be confused with the set of Heyting p-terms

$$\{\ \mathrm{a} : \Gamma \vdash \mathrm{a} : \mathrm{A}, \text{ for some context } \Gamma, \text{ and some A in } [\bot,\to,\wedge,\vee,\forall,\exists]\ \}.$$

The usual syntactic notions (*free/bound p-variables*, *subterm*, etc.), related to $\vdash$[**HQ**], are defined, *mutatis mutandis*, as for $\vdash_{(*,\&)}$[**CQ**], and/or $\vdash$[**MQ**]. A *Heyting proof-combinator* is a closed proof-term in $\vdash$[**HQ**]. Clearly, **HQ** $\Vdash$ A [i.e., A is provable in **HQ**], iff [ ] $\vdash$ a : A, in $\vdash$[**HQ**], for some Heyting proof-combinator a.

Where the family ω_A (A in $[\bot,\to,\wedge,\vee,\forall,\exists]$) and the "minimal"/Heyting $[\vee,\exists]_m$-proof-operations are defined as earlier, let $[\mathrm{a}]_\&$ be the definitional expansion, in $\vdash_\&$[**CQ**], of a Heyting proof-term a in $\vdash$[**HQ**], relative to these definitions. Analogously, for A in $[\bot,\to,\wedge,\vee,\forall,\exists]$, $[\mathrm{A}]_\&$ is supposed to expand A in $[\bot,\to,\wedge,\forall]$, *modulo* the standard (Boolean) definitions of $\vee$, $\exists$. One has first the following *definitional embedding* result.

Corollary ($\vdash$[**HQ**]-*definitional embedding into* $\vdash_\&$[**CQ**]). For all formulas A in $[\bot,\to,\wedge,\vee,\forall,\exists]$, any proof-context Γ and all Heyting proof-terms a of $\vdash$[**HQ**] (where $\vdash_h$ stands for derivability in $\vdash$[**HQ**]),

(1) $\Gamma \vdash_h \mathrm{a} : \mathrm{A} \Rightarrow \Gamma \vdash_\& [\mathrm{a}]_\& : [\mathrm{A}]_\&$, whence, in particular,
(2) **HQ** $\Vdash$ A $\Rightarrow$ [ ] $\vdash_\&$ c : $[\mathrm{A}]_\&$, for some *Boolean* proof-combinator c.

Proof. As for $\vdash$[**MQ**], while $(\to\mathrm{i}\omega)_h$ is derivable in $\vdash_{(\&)}$[**CQ**], with $\omega_\mathrm{A}(\mathrm{e}) := \gamma\mathrm{x}{:}\neg\mathrm{A.e}$ $[\mathrm{x} \notin \mathrm{FV}_\lambda(\mathrm{e})]$. □

The long and boring list of facts reviewed in the above says that $\lambda\gamma_\&$**CQ** *contains equationally* the Heyting proof-calculus λ**HQ**, as an *extension* of it, *modulo* an appropriate definitional embedding. We leave to the reader the task of displaying a *formal translation* (*embedding*) $[\ldots]_\& : \lambda\mathbf{HQ} \longrightarrow \lambda\gamma_\&\mathbf{CQ}$ that could make this official (the *theorem* to get reading this time: $\lambda\gamma_\&\mathbf{CQ} \supset [\lambda\mathbf{HQ}]_\&$). As in the case of λ**MQ** [*Minimalkalkül*], the extension is *proper*.

At last, worth recording separately is the *proof-consistency* of the Heyting first-order logic **HQ**:

Theorem (*Proof-consistency for* **HQ**). **Cons**(λ**HQ**).

Proof. Because $\lambda\gamma_{\&}\mathbf{CQ}$ extends $\lambda\mathbf{HQ}$ and we had $\mathbf{Cons}(\lambda\gamma_{\&}\mathbf{CQ})$. $\square$

One might also note the fact that the entire procedure leading to $\mathbf{Cons}(\lambda\mathbf{HQ})$ is also *admissible intuitionistically*, since the critical step in the previous proof of $\mathbf{Cons}(\lambda\gamma_{\&}\mathbf{CQ})$ is just a (*syntactic*) *translation* of $\lambda\gamma_{\&}\mathbf{CQ}$ into $\lambda\pi!$ (a proper fragment of the *Minimalkalkül*). So, *we have a translation of the Heyting proof-calculus* $\lambda\mathbf{HQ}$ *into a proper fragment of it* ($\lambda\pi!$).

In the end, *classical logic preserves* [*the proof-theory of*] **HQ** *entirely*. This is *as it should be*, since *any intuitionistically correct first-order* [**HQ**-] *proof*-[*identity*]-*principle must be also classically correct*.[24] As we have seen before, the converse statement is also true, *modulo* frequent uses of an appropriate dictionary, viz. the Glivenko "negative" translation, as applied to the *proofs themselves*.

[24]In particular, rephrasing the claim – "negatively" – in terms of the current ʾεπιστήμη, any atempt to a *theory of classical proofs* that fails to explain – among other things – the Heyting/intuitionistic *first-order* "proof-isomorphisms" is also *theoretically inadequate* – or else: it is *not about classical proofs*, either. The restrictive *proviso* to the "first-order" is relevant here. Genetically, the concept of a *first-order property* is a fall-out of semantic considerations that are *not intelligible* intuitionistically. It is, in fact, *rigorously meaningless* for the intuitionist, whose essential activity as a *paradigmatic mathematician* – activity that includes logic, as a reflection on this very activity – is naturally immersed into a *higher* [order] *medium*, so to speak. At this level, there are, of course, proof-principles (and so "proof-isomorphisms", perhaps) that are not compatible with what we use to call "classical mathematics" (cf., e.g., [Troelstra 80]). Whether the latter do still belong to the "province of logic" in some sense – as distinct from mathematics, and insafar the *distinguo* is feasible at all within intuitionism – is matter for much larger a debate. Contrast the above with William Tait's somewhat abrupt case "against intuitionism" in his [83] (where, the discussion – running in type-theoretic terms – ignores the rôle of the *negative properties* in Brouwer), and see, possibly, [Tait 86], for a philosophical complement [we were unable to identify the promised sequel in print]. Tait's views might well apply – up to a certain point – to Bishop's *constructive mathematics* (cf. [Bishop 67], [Bishop & Bridges 85], and [Beeson 85]), or to Martin-Löf's way of understanding it, for instance, but certainly *not* to Brouwer and Heyting... In this respect, such views are also – *mutatis mutandis* – typical for a widespread kind of [loose] talk about "intuitionistic logic", which makes, roughly, the Heyting [first-order] logic = *Minimalkalkül* + the *ex falso*-rule. The looseness hides conceptual confusion, although the latter remains invisible, as long as the proofs themselves – together with the appropriate isomorphisms – are not in the picture. In order to *understand* Brouwer we have to put them back. For those concerned with *theoretical* logic alone, this is the main issue in Brouwer's work.

Appendix

Assorted topics

[**1**] *A Gentzen L-system for* **CQ**. One can obtain *a Gentzen L-style proof-system* $L[\mathbf{CQ}]$ for first-order classical logic **CQ** – a "sequent calculus", with "sequents" that are "singular on the right" – by *forgetting the proof-information from* $\vdash_{(\&,*)}[\mathbf{CQ}]$. The outcome is, practically, a *standard* way of coping with "proof-theory" within the proof-theoretic tradition (post-Gentzen).

Write, e.g., $[\Gamma]$ for Γ_u,$A_1,\ldots,A_n$, if $\Gamma = \Gamma_u[x_1{:}A_1]\ldots[x_n{:}A_n]$ is a proof-context of $\vdash_{(\&,*)}[\mathbf{CQ}]$.

As ever, the notation "$\Gamma_u[\Gamma]$" means that the formulas of $[\Gamma]$ may contain free **U**-parameters from among those occurring in the list $\Gamma_u = [u_1{:}\mathbf{U}]\ldots[u_m{:}\mathbf{U}]$.)

With $\Vdash$ in place of $\vdash_{(\&,*)}$, one should have the following elliptic – "provability-only" – variants of the $\vdash_{(\&,*)}[\mathbf{CQ}]$ stratification (-rules):

1.1 "Structural" rules.

$< I \Vdash >$	$A \Vdash A$,	
$< K \Vdash >$	$[\Gamma] \Vdash C$	$\Rightarrow [\Gamma],A \Vdash C$,
$< KW \Vdash >$	$[\Gamma],A \Vdash C$	$\Rightarrow [\Gamma],A,A \Vdash C$,
$< (\$)W \Vdash >$	$[\Gamma],A,A \Vdash C$	$\Rightarrow [\Gamma],A \Vdash C$,
$< C \Vdash >$	$[\Gamma],A,B \Vdash C$	$\Rightarrow [\Gamma],B,A \Vdash C$,
$< K_u \Vdash >$	$[\Gamma] \Vdash C$	$\Rightarrow [\Gamma][u{:}\mathbf{U}] \Vdash C$,
$< KW_u \Vdash >$	$[\Gamma][u{:}\mathbf{U}] \Vdash C\llbracket u\rrbracket$	$\Rightarrow [\Gamma][u{:}\mathbf{U}][u{:}\mathbf{U}] \Vdash C\llbracket u\rrbracket$,
$< W_u \Vdash >$	$[\Gamma][u{:}\mathbf{U}][u{:}\mathbf{U}] \Vdash C\llbracket u\rrbracket$	$\Rightarrow [\Gamma][u{:}\mathbf{U}] \Vdash C\llbracket u\rrbracket$,
$< C_u \Vdash >$	$[\Gamma][u{:}\mathbf{U}][v{:}\mathbf{U}] \Vdash C\llbracket u,v\rrbracket$	$\Rightarrow [\Gamma][v{:}\mathbf{U}][u{:}\mathbf{U}] \Vdash C\llbracket u,v\rrbracket$.

1.2 "Cut"-rules.

$< \$K \Vdash >$	$[\Gamma],A \Vdash C$	$\Rightarrow [\Gamma] \Vdash C$, if C does not depend on A,
$< \$_uK \Vdash >$	$[\Gamma][u{:}\mathbf{U}] \Vdash C$	$\Rightarrow [\Gamma] \Vdash C$, if $u \notin FV_u(C)$,
$< \$_uW \Vdash >$	$[\Gamma][u{:}\mathbf{U}][v{:}\mathbf{U}] \Vdash C\llbracket u,v\rrbracket$	$\Rightarrow [\Gamma][u{:}\mathbf{U}][u{:}\mathbf{U}] \Vdash C\llbracket u,u\rrbracket$,
$< \$ \Vdash >$	$[\Gamma_1] \Vdash A$; $[\Gamma],A \Vdash C$	$\Rightarrow [\Gamma][\Gamma_1] \Vdash C$,
$< \$_{[u]} \Vdash >$	$\Gamma_u \Vdash \mathbf{t} :: \mathbf{U}$; $[\Gamma]\llbracket u{:}\mathbf{U}\rrbracket \Vdash C\llbracket u\rrbracket$	$\Rightarrow \Gamma_u[\Gamma\llbracket u{:=}\mathbf{t}\rrbracket] \Vdash C\llbracket u{:=}\mathbf{t}\rrbracket$.

2 "Inference" rules.

$(\rightarrow\Vdash)$	$[\Gamma],B \Vdash C$; $[\Gamma_1] \Vdash A$	$\Rightarrow [\Gamma],A\rightarrow B,[\Gamma_1] \Vdash C$,
$(\rightarrow\lambda\dashv\|)$	$[\Gamma],A \Vdash B$	$\Rightarrow [\Gamma] \Vdash A \rightarrow B$,
$(\rightarrow\gamma\dashv\|)$	$[\Gamma],\neg A \Vdash \bot$	$\Rightarrow [\Gamma] \Vdash A$,
$(\wedge\Vdash)$	$[\Gamma]A,B \Vdash C$	$\Rightarrow [\Gamma],A\wedge B \Vdash C$,
$(\wedge\dashv\|)$	$[\Gamma_1] \Vdash A$; $[\Gamma_2] \Vdash B$	$\Rightarrow [\Gamma_1][\Gamma_2] \Vdash A \wedge B$,
$(\vee\Vdash)$	$[\Gamma_1],A,\neg C \Vdash \bot$; $[\Gamma_1],B,\neg C \Vdash \bot$	$\Rightarrow [\Gamma_1][\Gamma_2],A\vee B \Vdash C$,
$(\vee\dashv\|)$	$[\Gamma],\neg A,\neg B \Vdash \bot$	$\Rightarrow [\Gamma] \Vdash A \vee B$,
$(\forall\Vdash)$	$\Gamma_u \Vdash \mathbf{t} :: \mathbf{U}$; $\Gamma_u[\Gamma],A\llbracket\mathbf{t}\rrbracket \Vdash C$	$\Rightarrow \Gamma_u[\Gamma],\forall u.A\llbracket u\rrbracket \Vdash C$,
$(\forall\dashv\|)$	$[\Gamma][u{:}\mathbf{U}] \Vdash A\llbracket u\rrbracket$	$\Rightarrow [\Gamma] \Vdash \forall u.A\llbracket u\rrbracket$,
$(\exists\Vdash)$	$[\Gamma][u{:}\mathbf{U}],A\llbracket u\rrbracket,\neg C \Vdash \bot$,	$\Rightarrow [\Gamma],\exists u.A\llbracket u\rrbracket \Vdash C$, **where** $u \notin FV_u(C)$,
$(\exists\dashv\|)$	$\Gamma_u \Vdash \mathbf{t} :: \mathbf{U}$; $\Gamma_u[\Gamma],\neg A\llbracket\mathbf{t}\rrbracket \Vdash \bot$	$\Rightarrow \Gamma_u[\Gamma] \Vdash \exists u.A\llbracket u\rrbracket$.

Here, the notation "$[\Gamma][u{:}\mathbf{U}]$" means that the **U**-parameter u does not occur in the formulas of $[\Gamma]$ and, as for $\vdash_{(\&,*)}[\mathbf{CQ}]$, if Γ_u is a list of **U**-parameters, the notation "$\Gamma_u \Vdash \mathbf{t} :: \mathbf{U}$" means that the **U**-term $\mathbf{t}$ contains possibly free **U**-variables from Γ_u.

Without a proper proof-notation, the $\vdash_{(\&,*)}$[**CQ**]-rules $< W >$, $< \$W >$ are identified, while $< \$K >$ cannot be even stated in formal terms. Further, $< KW \Vdash >$ and $< KW_u \Vdash >$ can be viewed as instances of $< K \Vdash >$ and $< K_u \Vdash >$, resp.

In elliptic variant, the **U**-rules $< K_u >$, $< KW_u >$, $< W_u >$, $< C_u >$ and $< \$_u K >$, $< \$_u W >$, $< \$_{[u]} >$ appear as mere pedantic stipulations about the use of free "individual" variables in $\Vdash$-"derivations". In the same spirit, $< I \Vdash >$ should have been stated rather as: Γ_u,A $\Vdash$ A, where $\mathrm{FV}_u(\mathrm{A}) = \{\mathrm{u}_1,\ldots \mathrm{u}_m\}$, with $\Gamma_u = [\mathrm{u}_1{:}\mathbf{U}]\ldots[\mathrm{u}_m{:}\mathbf{U}]$, say. As expected, $< \$_u W \Vdash >$ can be obtained from $< \$_{[u]} \Vdash >$.

As is well-known, a general Gentzen "sequent" $\mathrm{C}_1,\ldots,\mathrm{C}_p \Vdash \mathrm{D}_1,\ldots,\mathrm{D}_q$, can be viewed as a shorthand for an appropriate "right-singular sequent" $\mathrm{A}_1,\ldots,\mathrm{A}_n \Vdash \mathrm{B}$ (= the *elliptic* form of a proof-statement

$$[\mathrm{u}_1\ldots \mathrm{u}_m][\mathrm{x}_1{:}\mathrm{A}_1]\ldots[\mathrm{x}_n{:}\mathrm{A}_n] \vdash \mathrm{b}[\![\mathrm{u}_1\ldots \mathrm{u}_m,\mathrm{x}_1\ldots \mathrm{x}_n]\!] : \mathrm{B}[\![\mathrm{u}_1\ldots \mathrm{u}_m]\!]).$$

In other words, one can use *systematic abbreviations* [Prawitz 65]

$(\Vdash^L)$ $[\Gamma] \Vdash^L \mathrm{A}_1, \ldots, \mathrm{A}_n \equiv_{df} [\Gamma],\neg\mathrm{A}_1,\ldots,\neg\mathrm{A}_n \Vdash \bot$,

in order to obtain familiar Gentzen "sequent"-like rules for $\vee$, $\exists$:

$(\vee\Vdash^L)$ $[\Gamma_1]$,A $\Vdash^L$ C; $[\Gamma_2]$,B $\Vdash^L$ C $\Rightarrow [\Gamma_1][\Gamma_2]$,A$\vee$B $\Vdash^L$ C,
$(\vee\text{–}\Vdash^L)$ $[\Gamma] \Vdash^L$ A, B $\Rightarrow [\Gamma] \Vdash^L$ A $\vee$ B,
$(\exists\Vdash^L)$ $[\Gamma][\mathrm{u}{:}\mathbf{U}]$,A$[\![\mathrm{u}]\!] \Vdash^L$ C $\Rightarrow [\Gamma]$,$\exists$u.A$[\![\mathrm{u}]\!] \Vdash^L$ C, **where** u $\notin \mathrm{FV}_u(\mathrm{C})$,
$(\exists\text{–}\Vdash^L)$ $\Gamma_u \Vdash \mathbf{t} :: \mathbf{U}$; $\Gamma_u[\Gamma] \Vdash^L \mathrm{A}[\![\mathbf{t}]\!] \Rightarrow \Gamma_u[\Gamma] \Vdash^L \exists\mathrm{u.A}[\![\mathrm{u}]\!]$,

as a shorthand for the previous $(\vee\Vdash)$, $(\vee\text{–}\Vdash)$, $(\exists\Vdash)$, $(\exists\text{–}\Vdash)$, resp.

Of course, this makes $(\rightarrow\gamma\text{–}\Vdash^L)$, i.e., the elliptic variant of $(\rightarrow i\gamma)$ [*reductio ad absurdum*], look uninteresting $(\Gamma \Vdash^L \mathrm{A} \Rightarrow \Gamma \Vdash^L \mathrm{A})$; its genuinely classical effect is, however, already incorporated into the "definition" $(\Vdash^L)$ of a "right-multiple sequent".

In particular, if understood in terms of $\Vdash^L$-abbreviations, the so-called rule of "contraction on the right":

$< W \text{–}\Vdash^L >$ $[\Gamma_1] \Vdash^L [\Gamma_2]$,A,A $\Rightarrow [\Gamma_1] \Vdash^L [\Gamma_2]$,A,

is *not a "structural" rule*, but rather *an ambiguous representation of distinct proof-operators* [= *derivation rules*], as, e.g.,

$\varepsilon([\mathrm{x}{:}\neg\mathrm{A}].\mathrm{a}[\![\mathrm{x}]\!]{:}\mathrm{A}) = \gamma([\mathrm{x}{:}\neg\mathrm{A}].\mathrm{x}(\mathrm{a}[\![\mathrm{x}]\!]){:}\bot) : \mathrm{A}$,
$\gamma([\mathrm{z}{:}\neg\mathrm{A}].\mathrm{e}[\![\mathrm{z,z}]\!]{:}\bot) = \gamma([\mathrm{x}{:}\neg\mathrm{A}].\mathrm{x}(\gamma([\mathrm{y}{:}\neg\mathrm{A}].\mathrm{e}[\![\mathrm{x,y}]\!]{:}\bot)){:}\bot) : \mathrm{A}$,

etc. [*modulo* proof-equality in, e.g., $\lambda\gamma_{(\&)}$**CQ**], where the corresponding [*non-elliptic*] proof-situations are

$\Gamma[\mathrm{x}{:}\neg\mathrm{A}] \vdash \mathrm{a}[\![\mathrm{x}]\!] : \mathrm{A} \Rightarrow \Gamma \vdash \varepsilon\mathrm{x}{:}\neg\mathrm{A.a}[\![\mathrm{x}]\!] \equiv_{df} \gamma\mathrm{x}{:}\neg\mathrm{A.x}(\mathrm{a}[\![\mathrm{x}]\!]) : \mathrm{A}$,
$\Gamma[\mathrm{x}{:}\neg\mathrm{A}][\mathrm{y}{:}\neg\mathrm{A}] \vdash \mathrm{e}[\![\mathrm{x,y}]\!] : \bot \Rightarrow \Gamma \vdash \gamma\mathrm{x}{:}\neg\mathrm{A.x}(\gamma\mathrm{y.e}[\![\mathrm{x,y}]\!]) = \gamma\mathrm{z}{:}\neg\mathrm{A.e}[\![\mathrm{z,z}]\!] : \mathrm{A}$,

resp., for $[\Gamma] = [\Gamma_1][\Gamma_2^-]$, with $[\Gamma_2^-] = \neg\mathrm{B}_1,\ldots,\neg\mathrm{B}_n$, if $[\Gamma_2] = \mathrm{B}_1,\ldots,\mathrm{B}_n$.

As a consequence, any attempt to define a "notion" of *proof-reduction* and/or *proof-equality* for **CQ** by direct transformations applied to general ["right multiple"] *L*-sequents rests, inevitably, on *conceptual confusion*.[25]

The resulting *L*[**CQ**]-system has *no real proof-theoretic import* in the present setting, except perhaps for the fact that the proof-discarding procedure used to generate it yields a straightforward way of showing *provability completeness for* $\vdash_{(\&,*)}$[**CQ**], i.e.,

[25]Would-be "invariants" so obtained might "characterize" some particular way of organizing a preferred *visual display*, but *not* proof-behaviors. With a technical – philosophical – term, this would also amount to a *category mistake*. The above make obvious, we hope, the fact that the use of the *primitive* "notion" of a general ["right-multiple"] "sequent" in the description of **CQ**-proofs ("derivations") is *a way of mishandling the subject-matter of logic*. Otherwise, the wisdom resulting from "eliminating the cuts" is available, in *a proper theoretical setting*, by specific metatheorems, as, e.g., the *Sub-proof Theorem*, the *Unicity of Typing* for Boolean proof-terms (the *Proof-categoricity* property), and – relative to specific Boolean equational theories or reduction systems – *Proof-consistency*, [*Strong*] *Normalization* properties, etc. [Yet, qualitatively, the latter type of information makes up a *structured* body of mathematical knowledge and belongs, of course, to a different order of things. . .]

$$\mathbf{CQ} \Vdash \mathrm{A} \iff [\,] \vdash_{(\&,*)} \mathrm{a} : \mathrm{A}, \text{ for some Boolean proof-combinator a.}$$

This information is, however, available directly, by comparing, e.g., the $\vdash_{\&}[\mathbf{CQ}]$-rules with an appropriate N-formulation for **CQ** [Prawitz 65] (recall that $\vdash[\mathbf{CQ}]$, $\vdash_{\&}[\mathbf{CQ}]$, and $\vdash_{*}[\mathbf{CQ}]$ are stratification-equivalent).

As an *exercise*, the reader could try to obtain an analogous L-style formulation $L[\mathbf{HQ}]$ for the Heyting first-order logic **HQ**. (*Hints*. Use $(\vee\Vdash^{L})$, $(\exists\Vdash^{L})$, $(\exists\dashv\!\Vert^{L})$, in place of $(\vee\Vdash)$, $(\exists\Vdash)$, $(\exists\dashv\!\Vert)$, resp., specialize $(\rightarrow\gamma\dashv\!\Vert)$ to an appropriate instance $(\rightarrow\omega\dashv\!\Vert)$, *via* $< \$K \Vdash >$, where "$\bot$ does not depend on $\neg$A", and split $(\vee\dashv\!\Vert^{L})$ into cases $(\vee_i\dashv\!\Vert^{L})$, [i := 1,2]. **NB**: Consider the instances $\omega_{\mathrm{A}}(\mathrm{e}{:}\bot) \equiv \gamma\mathrm{x}{:}\neg\mathrm{A.e}$ of γ-abstraction, with x not free in e, realizing also the fact that the *Minimalkalkül* $[\vee,\exists]$-proof-operators are identical with the Heyting $[\vee,\exists]$-proof-operators and that they record actually the "positive contents" of the Boolean $[\vee,\exists]$-proof-operators.)

[**2**] *Proof-transformations in* $\lambda\gamma_{(\&)}\mathbf{CQ}$. As expected, a *standard* notion of *proof-reduction* ["proof-détour elimination"] $\rhd\!\!-$ *for* $\lambda\gamma_{(\&)}\mathbf{CQ}$ can be obtained from the $\lambda\gamma_{(\&)}\mathbf{CQ}$-"axiomatics" above, by orienting the equality-rules [left to right] and by leaving out the symmetry condition [σ]. (In common λ-calculus parlance, $\rhd\!\!-$ is a "strong" notion, i.e., it is *extensional*.)

This induces a natural concept of a *normal proof* ["stable $\lambda\gamma$-form"] in $\lambda\gamma_{(\&)}\mathbf{CQ}$ (see [Rezuş 90] for details about the properties of proof-reduction in a classical proof-calculus $\lambda\gamma(\pi)!$ which is, essentially, the same thing as $\lambda\gamma_{(\&)}\mathbf{CQ}$ without γ-diagonalization).

Where $\mathbf{CQ} \Vdash$ A, let $\sum$(A) be the set of normal proofs of A in $\lambda\gamma_{(\&)}\mathbf{CQ}$ (referred to here as *the normal proof-spectrum of* A). In $\lambda\gamma_{(\&)}\mathbf{CQ}$ we have – "principially", so to speak – no means of establishing significant relations *inside* normal proof-spectra. In particular, we cannot compute a normal proof a_1 of A from another one $\mathrm{a}_2 \in \sum$(A).

In order to "exhaust" normal proof-spectra, one can use, however, *type-preserving computation* rules that are different in character from those identified previously as *reduction* [= "détour elimination"] rules. The latter are meant to define a special form of expansion of proof-terms, allowing us to "link" *effectively* distinct normal forms in a given proof-spectrum.

Examples. The following proof-term transformations should very likely suffice for $\lambda\gamma\mathbf{CQ}$, to this purpose :

$\{\succ\}$ $\Gamma \vdash \mathrm{f} \succ \mathrm{g}$ [: C], if $\Gamma \vdash \mathrm{f} \rhd\!\!- \mathrm{g} : \mathrm{C}$,
$\{\beta\lambda\}$ $\Gamma \vdash \mathrm{f}(\lambda\mathrm{x{:}A.e}[\![\mathrm{x}]\!]) \succ \gamma\mathrm{y}{:}\neg\mathrm{B.e}[\![\mathrm{x}{:=}\gamma\mathrm{z}{:}\neg\mathrm{A.y(f(z))}]\!]$ [: B], if $\Gamma \vdash \mathrm{f} : \neg\mathrm{A} \rightarrow \mathrm{B}$ and $\Gamma[\mathrm{x{:}A}] \vdash \mathrm{e}[\![\mathrm{x}]\!] : \bot$,
$\{\beta\gamma\}$ $\Gamma \vdash \mathrm{f}(\gamma\mathrm{x}{:}\neg\mathrm{A.e}[\![\mathrm{x}]\!]) \succ \gamma\mathrm{y}{:}\neg\mathrm{B.e}[\![\mathrm{x}{:=}\lambda\mathrm{z}{:}\mathrm{A.y(f(z))}]\!]$ [: B], if $\Gamma \vdash \mathrm{f} : \mathrm{A} \rightarrow \mathrm{B}$ and $\Gamma[\mathrm{x}{:}\neg\mathrm{A}] \vdash \mathrm{e}[\![\mathrm{x}]\!] : \bot$.

Obviously, the computation rules $\{\beta\lambda\}$ and $\{\beta\gamma\}$ above are type-[= provability]-preserving.

By the $\succ$-variant of $[\beta\gamma\bot]$ (and $[\eta \rightarrow \lambda]$, a characteristic feature of $\rhd\!\!-$), one has immediately the special cases:

$\{\beta\lambda\bot\}$ $\Gamma \vdash \mathrm{f}(\lambda\mathrm{x{:}A.e}[\![\mathrm{x}]\!]) \succ \mathrm{e}[\![\mathrm{x}{:=}\gamma\mathrm{z}{:}\neg\mathrm{A.f(z)}]\!]$ [: $\bot$], if $\Gamma \vdash \mathrm{f} : \neg\neg\mathrm{A}$ and $\Gamma[\mathrm{x{:}A}] \vdash \mathrm{e}[\![\mathrm{x}]\!] : \bot$,
$\{\beta\gamma\bot\}$ $\Gamma \vdash \mathrm{f}(\gamma\mathrm{x}{:}\neg\mathrm{A.e}[\![\mathrm{x}]\!]) \succ \mathrm{e}[\![\mathrm{x}{:=}\lambda\mathrm{z}{:}\neg\mathrm{A.f(z)}]\!] \succ \mathrm{e}[\![\mathrm{x}{:=}\mathrm{f}]\!]$ [: $\bot$], if $\Gamma \vdash \mathrm{f} : \neg\mathrm{A}$ and $\Gamma[\mathrm{x}{:}\neg\mathrm{A}] \vdash \mathrm{e}[\![\mathrm{x}]\!] : \bot$.

Where $\prec$ is the converse of $\succ$, it is easy to see that, e.g., $\{\beta\gamma\bot\}$ yields, for all formulas A (in $[\bot,\rightarrow,(\forall)]$, say), and for any two proof-terms a_1, a_2 (in $\lambda\gamma\mathbf{CQ}$ and, even, in its sub-system without γ-diagonalization),

$(\rightarrowtail\!\!\prec)$ $\Gamma \vdash \mathrm{a}_1, \mathrm{a}_2 : \mathrm{A} \Rightarrow \Gamma \vdash \mathrm{a}_1 \prec \mathrm{a} \succ \mathrm{a}_2 : \mathrm{A}$, for some a,

(that is, ultimately, we have "*proof-irrelevance*" relative to the symmetric closure, $\rightarrowtail\!\!\prec$, say, of $\succ$, whence $\rightarrowtail\!\!\prec$ is rather uninteresting theoretically).

From an intuitive point of view, the rules $\{\beta\lambda\}$, $\{\beta\gamma\}$ have the effect of "moving around" particular applications of *reductio ad absurdum* within a proof (-term). Formally, one has a *rigorous counterpart* of the concept of a *reductio-transfer*.

A careful manipulation of this notion leads to *unexpected* [oft useful] observations. One can establish, for instance, the fact that *the normal proof-spectra of* **CQ**-*provable formulas can be exhausted by a systematic use of reductio-transfer* (where the "systematic use" has, ultimately, the character of a *Beth-tree* [-like] *proof-search for* **CQ**).

[**3**] *"Double negation" interpretations of classical proofs: a theme from* [Kolmogorov 25]. The Glivenko [28,29] translation maps **CQ** into (a fragment of) **MQ**. By [Kolmogorov 25] we know that **CQ** can be also translated into **MQ** (and/or **HQ**), using – in an essential way – specific operations of "double-negation introduction".

As regards provability, the basic idea consists of performing a *uniform* "double negation" interpretation of **CQ**-formulas within the **HQ**-syntax; this is done recursively on subformulas, such that, ultimately, the "primes" are negated at least once during the process. In the limit case (Kolmogorov), if we insert double negations in front of *every* subformula of a classically valid formula – *outermost* pair included –, the result is a formula that is provable in the Heyting logic.

There are several variations on this theme: all of them yield an analogous (*provability*) variant of the first "Glivenko Lemma", – applying, *mutatis mutandis*, to **CQ** and **HQ** – although they are not informative equationally. The main reason is in the fact that the straightforward extension of a "double negation" translation to proof-terms does *not* preserve the equality of the ordinary (*extensional*) typed λ-calculus, λ^{τ}.[26]

We discuss next, briefly, the proof-extension of the original Kolmogorov [25] translation-pattern. In order to evidentiate the main idea, we consider only typed languages based on a [$\perp$,$\rightarrow$]-type-structure (otherwise, once the basics are understood, supplying the missing cases is a relatively trivial affair).

The general situation can be captured informally, at a provability level, by noting that

(a) where $\neg\neg(\dots) \equiv ((\dots) \rightarrow \perp \rightarrow \perp)$, a "double negation" translation – *from* **CQ** *to* **CQ** [*sic*!] – is supposed to insert *uniformly* $\neg\neg(\dots)$-"contexts" around *some* sub-types in a [$\perp$,$\rightarrow$,...]-type, and that,

(b) in order to realize the intended meaning of the translation [the target should be, in the end, *at most* **HQ**, not **CQ**, in general], the "negative" occurrences of the "primes" must fall within the scope of a $((\dots) \rightarrow \perp)$-"context" [i.e., they must be negated], once the process is completed.

Would-be variations occur by *specifying the exact distribution of the* $\neg\neg$-*insertions*. If the translation is also recursive (as in the familiar cases, encountered in the literature), we have some control on the typology of the variants. Ignoring for a moment the translation $(\dots)^a$ of the atoms ($\perp$ and the "primes"), we have *prima facie*, with a primitive [$\perp$,$\rightarrow$]-structure, three kinds of possibly distinct "double negation" translations $(\dots)^k$, of the Kolmogorov-kind. If C $\equiv$ (A $\rightarrow$ B), and A, B are atoms, we can get *either*

[26] As a matter of fact, this depends also on the choice of type-primitives. The characterization above applies *verbatim* to situations where the source language has at least *primitive* $\rightarrow$-types: [Kolmogorov 25] is a case in point. If the $\rightarrow$-types must be also defined ["simulated"] in terms of other primitives –, e.g., if we rely on [$\wedge$,$\neg$,($\forall$)]-formulations of classical logic, as in [Gödel 33], say – there are additional complications, although the outcome is the same. Somewhat *a priori*, in such cases we can *never* retrieve the familiar *extensionality principles* of λ-calculus, on different reasons, though. There is some interest in this type of *anomaly*, because equational systems (or yet, in general, reduction systems) induced by a "double negation"-interpretation *make sense from a computational point of view*. *Mutatis mutandis*, such situations give rise to weak proof-systems for classical logic, falling under a would-be general rubric *intensional proof-theories*. This area of investigation has not beeen suitably charted thus far: at the time of writing, we have at hand, roughly, a body of *closely related examples* and an indefinite number of (unanswered) *questions* about them. Historically, a translation much similar to Gödel's has been found by Gentzen (1933), independently (in view of [Gödel 33], he didn't estimate it interesting enough to deserve publication, cf. [Gentzen 74]). On a pure provability level, the theme has been oft revisited (S. Kuroda 1951, J. Łukasiewicz 1952, M. H. Löb 1976, H. Friedman 1978, D. Leivant 1985, etc.). There is no real theoretical profit in examining separately the equational proof-behaviors induced by the proposed mappings (one can retrieve them by proceeding systematically, anyway).

(1^k) $(C)^k \equiv \neg\neg(\neg\neg(A^a) \to \neg\neg(B^a))$, (as in [Kolmogorov 25]), *or*
(2^k) $(C)^k \equiv \neg\neg((A^a) \to \neg\neg(B^a))$, *or*
(3^k) $(C)^k \equiv \neg\neg(\neg\neg(A^a) \to (B^a))$.

Of these, the third choice is *non-productive for the intended purposes*, although – technically – it yields a perfect endomorphism of **CQ**-proofs (in other words, the resulting proof-term mapping won't eliminate the genuinely Boolean uses of γ's). The second one respects the Kolmogorov translation-pattern, although it is, in a sense, worse than the original Kolmogorov proposal: the *intended* proof-extensions do not even preserve [all instances of] the usual β-equality.[27]

Let, as ever, $\lambda\gamma\oint$ be the ([⊥,→])-fragment of $\lambda\gamma_{(\&,*)}$**CQ**. [This is a system with *full* γ-diagonalization.] (In order to ease readability, we assume that the notations $(\ldots)^K$, $(\ldots)_K$ bind stronger than any other syntactic operation.) In terms of proofs, the Kolmogorov [25] "double negation" translation reads as follows:

Definition (*The Kolmogorov* [25] *"negative" translation*).

Define a map $(\ldots)^K$ from proof-statements $\Gamma \vdash a : A$ of $\lambda\gamma\oint$ to ([⊥,→,...]) proof-statements of the form $(\Gamma)^K \vdash (a)^K : (A)^K$ [in λ**MQ**], λ**HQ**], by:

- $(\mathbf{t})^K \equiv \mathbf{t}$, for any **U**-term **t**,
- $(\bot)^K \equiv \neg\neg(\bot_K)$ [$\equiv \neg\neg(\bot)$], **where** $\bot_K \equiv \bot$,
- $(A)^K \equiv \neg\neg(A_K)$ [$\equiv \neg\neg(A)$], for any "prime" A, **where** $A_K \equiv A$,
- $(A \to B)^K \equiv \neg\neg(A \to B)_K$ [$\equiv \neg\neg(A^K \to B^K)$], **where** $(A \to B)_K \equiv (A^K \to B^K)$,
- $(\Gamma)^K \equiv \Gamma_u \smile [x_1{:}(A_1)^K]\ldots[x_n{:}(A_n)^K]$, for any proof-context $\Gamma := \Gamma_u \smile [x_1{:}A_1]\ldots[x_n{:}A_n]$
- $(a)^K$, by induction on the structure of a, [with, as above, $(C)^K \equiv \neg\neg(C_K)$],
  - $(x)^K \equiv x$,
  - $(\lambda x{:}A.b[\![x]\!])^K \equiv \lambda k{:}\neg(A{\to}B)_K.k(\lambda x{:}A^K.(b[\![x]\!])^K)$,
  - $(fa)^K \equiv \lambda k{:}\neg B_K.(f)^K(\lambda x{:}(A{\to}B)_K.x(a)^K(k))$,
  - $(\gamma z{:}\neg A.e[\![z]\!])^K \equiv \lambda k{:}\neg A_K.(e[\![z]\!])^K[\![z{:=}\varphi_K(k)]\!](\Omega_K)$, **where**
    $\varphi_K(k) \equiv \lambda x{:}\neg(A{\to}\bot)_K.x(\lambda y{:}\neg\neg(A)_K.\lambda i{:}\top.y(k))$, and
    $\Omega_K \equiv \Omega$ [$\equiv \lambda x{:}\bot.x$].

The reader will check the fact that $(\ldots)^K$ is well-defined as a map. The extension of $(\ldots)^K$ to $\vdash_{(\&,*)}$[**CQ**] is straightforward (yet, variants are possible for the first-order case) [*exercise*]. We obtain the expected "*Kolmogorov Provability Lemma*", by applying $(\ldots)^K$ to **C(Q)**-proof-statements:

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[27] It is, nevertheless, unobjectionable on a provability level. As an aside, (2^k) occurs also naturally in considerations on the semantics of *imperative* programming languages, where it can be encountered under the label "*cps* [= *continuation passing style*] *translation*". The observation is due, independently, to Matthias Felleisen (in a type-free setting: Ph D Diss., Indiana Univ., Bloomington IN, 1987), Bruce Duba and Timothy Griffin (1990). Under the specific computational interpretation, the genuinely Boolean γ-constructs are identified as *non-local control*-phenomena (simplifying a bit, these are *jump*-like programming constructs). Originally, Felleisen and his associates used a mixed *combinatory reduction system* – in the sense of Jan Willem Klop –, based on a λ-calculus syntax with additional combinators *C* (*"control"*) and *A* (*"abort"*), resp. corresponding to *unstratified* versions of $\Delta[\![A]\!] \equiv_{df} \lambda x{:}\neg\neg A.\gamma y{:}\neg A.x(y))$ [*duplex negatio affirmat*], and $\omega[\![A]\!] \equiv_{df} \lambda x{:}\bot.\gamma y{:}\neg A.x$ [*ex falso quodlibet*], resp. [The latter one wasn't, in fact, necessary, since it is definable, if the former one is present.] As expected, the transcription of the results in $\lambda\gamma$-terms is straightforward. The – proof-theoretically offending – situation that the $\beta\eta$-rules fail to obtain, in general, is accommodated under the computational reading by using a so-called *call-by-value* λ-calculus instead. In particular, (1^k) and (2^k) above match, *mutatis mutandis*, after stratification, the well-known (Gordon D. Plotkin 1975) translations from *call-by-name*-style to *call-by-value*-style evaluation and conversely. This looks natural, indeed, in a pure computational setting, but we have – unfortunately – no reasonable proof-counterpart of this kind of reading the rules. More recent speculations on the theme "extracting computational content from classical proofs" [no relation to Bishop's ideas of the sixties] are by the way, from the present point of view. [Felleisen himself did – wisely – refrain from making hurried extrapolations; indeed, both *proving*- and *computing*-phenomena might well have *common roots*: the fact is that we don't have yet the right conceptual means of understanding what is going on.] We are indebted to Jon Seldin for signalling to us the work of Felleisen and Griffin (by Fall 1990), and to Matthias Felleisen, Tim Griffin, and Chet Murthy for further details (1990-1991).

Lemma (A. N. Kolmogorov, 1925). $\Gamma \vdash$ a : A [in $\lambda\gamma\oint$] $\Rightarrow$ $(\Gamma)^K \vdash (\mathrm{a})^K : (\mathrm{A})^K$ [in λ**(M,H)Q**].

Proof. By induction on the relevant (here: [$\bot$,$\rightarrow$]-) fragment of $\vdash_0$[**CQ**]. □

Remark ("*The Kolmogorov $\lambda\gamma$-calculus*"). It is relatively easy to check the fact that the $\lambda\gamma_{\&}$-equalities $[\beta \rightarrow \lambda]$, $[\eta \rightarrow \gamma]$, $[\beta\gamma \rightarrow]$, and $[\oint_0 \gamma]$ (i.e., the "weak diagonalization") are verified under the $(\dots)^K$-translation [*exercise*]. It is *not* so, however, for $[\eta \rightarrow \lambda]$, $[\beta\gamma\bot]$, and $[\oint\gamma]$ ("full diagonalization"). There is nothing we can do about the first one (as observed earlier, the absence of the usual extensionality principles is a general feature of this kind of approach). In place of the latter two, we have also a weaker – γ-"diagonal", so to speak – form of $[\beta\gamma\bot]$, viz.,

$$[\beta\gamma_0\bot]\ \ \Gamma \vdash \gamma\mathrm{x}{:}\neg\mathrm{A}.\gamma\mathrm{y}{:}\top.\mathrm{e}[\![\mathrm{x,y}]\!] = \gamma\mathrm{x}{:}\neg\mathrm{A}.\mathrm{e}[\![\mathrm{x,y}]\!][\![\mathrm{y}{:=}\lambda\mathrm{z}{:}\bot.\omega_\bot(\mathrm{z})]\!]\ [:\ \mathrm{A}],\ \ \text{if } \Gamma[\mathrm{x}{:}\neg\mathrm{A}][\mathrm{y}{:}\top] \vdash \mathrm{e}[\![\mathrm{x,y}]\!] : \bot.$$

([Kolmogorov 25] has, in fact, a primitive $\neg$ in place of $\bot$, while the original mapping would have required $\bot^K \equiv \bot$, with, perhaps, an additional "type-isomorphism" $\neg\neg(\bot) \equiv \bot$. The variant above simplifies the matter: it actually maps **C(Q)**-proof-statements [in]to λ**M(Q)**.)

From this, we obtain a somewhat *ad hoc* Boolean proof-theory (in the technical sense of the present notes), "*a Kolmogorov* [*typed*] *$\lambda\gamma$-calculus*", $\lambda\gamma_K$, say. As noticed in the above, the outcome is *an intensional proof-theory* [a sub-system of $\lambda\gamma\oint$, in fact]. As an additional idiosyncrasy (beyond the absence of the usual η-rule), by the failure of $[\beta\gamma\bot]$, $\lambda\gamma_K$ is unable to eliminate entirely "complex" [in favor of "prime"-only] uses of γ.

We don't know if the equational system determined by the list above is *complete* relative to the $(\dots)^K$-translation, nor if it has, in itself, any interesting (meta-theoretic) properties.

The second Kolmogorov translation-pattern (2^k) above admits of at least two different extensions to p-terms: the specific choice depends on a decision involving an additional parameter, viz. the *order of evaluating* p-terms in the source language. (As one would expect, the alternatives should be equivalent in some sense.) We write conveniently $(\dots)^+$, $(\dots)^-$ for the corresponding mappings, and $(\dots)^\pm$ for both.

Definition (*The "continuation-passing-style" translations*).

Define maps $(\dots)^\pm$ from proof-statements $\Gamma \vdash$ a : A of $\lambda\gamma\oint$ to ([$\bot$,$\rightarrow$,...]) proof-statements of the form $(\Gamma)^\pm \vdash (\mathrm{a})^\pm : (\mathrm{A})^\pm$ [in λ**HQ**], by:

- $(\mathbf{t})^\pm \equiv \mathbf{t}$, for any **U**-term **t**,
- $(\bot)^\pm \equiv \bot$,
- $(\mathrm{A})^\pm \equiv \mathrm{A}$, for any "prime" A,
- $(\mathrm{A} \rightarrow \mathrm{B})^\pm \equiv (\mathrm{A})^\pm \rightarrow \neg\neg((\mathrm{B})^\pm)$,
- $(\Gamma)^\pm \equiv \Gamma_u \smile [\mathrm{x}_1{:}(\mathrm{A}_1)^\pm]\dots[\mathrm{x}_n{:}(\mathrm{A}_n)^\pm]$, for any proof-context $\Gamma := \Gamma_u \smile [\mathrm{x}_1{:}\mathrm{A}_1]\dots[\mathrm{x}_n{:}\mathrm{A}_n]$
- $(\mathrm{a})^\pm$, by induction on the structure of a,
  - $(\mathrm{x})^\pm \equiv \lambda\mathrm{k}{:}\neg(\mathrm{A})^\pm.\mathrm{k}(\mathrm{x})$, if $\Gamma \vdash$ x : A, in $\lambda\gamma\oint$,
  - $(\lambda\mathrm{x}{:}\mathrm{A}.\mathrm{b}[\![\mathrm{x}]\!])^\pm \equiv \lambda\mathrm{k}{:}\neg(\mathrm{A}{\rightarrow}\mathrm{B})^\pm.\mathrm{k}(\lambda\mathrm{x}{:}(\mathrm{A})^\pm.(\mathrm{b}[\![\mathrm{x}]\!])^\pm)$,
  - $(\mathrm{fa})^+ \equiv \lambda\mathrm{k}{:}\neg(\mathrm{B})^+.(\mathrm{f})^+(\lambda\mathrm{x}{:}(\mathrm{A}{\rightarrow}\mathrm{B})^+.(\mathrm{a})^+(\lambda\mathrm{y}{:}(\mathrm{A})^+.\mathrm{x}(\mathrm{y})(\mathrm{k})))$,
  - $(\mathrm{fa})^- \equiv \lambda\mathrm{k}{:}\neg(\mathrm{B})^-.(\mathrm{a})^-(\lambda\mathrm{y}{:}(\mathrm{A})^-.(\mathrm{f})^-(\lambda\mathrm{x}{:}(\mathrm{A}{\rightarrow}\mathrm{B})^-.\mathrm{x}(\mathrm{y})(\mathrm{k})))$,
  - $(\gamma\mathrm{z}{:}\neg\mathrm{A}.\mathrm{e}[\![\mathrm{z}]\!])^\pm \equiv \lambda\mathrm{k}{:}\neg(\mathrm{A})^\pm.(\mathrm{e}[\![\mathrm{z}]\!])^\pm[\![\mathrm{z}{:=}\varphi_\pm(\mathrm{k})]\!](\Omega_\pm)$, **where** $\varphi_\pm(\mathrm{k}) \equiv \lambda\mathrm{x}{:}(\mathrm{A})^\pm.\lambda\mathrm{i}{:}\top.\mathrm{k}(\mathrm{x})$, and $\Omega_\pm \equiv \lambda\mathrm{x}{:}\bot.\omega_\bot(\mathrm{x})$.

We have immediately the following provability result (where, to our knowledge, the $(\dots)^-$-part has been first noticed by C. R. Murthy 1991):

Lemma (B. Duba, T. G. Griffin, 1990). $\Gamma \vdash$ a : A in $\lambda\gamma\oint \Rightarrow (\Gamma)^\pm \vdash (\mathrm{a})^\pm : \neg\neg(\mathrm{A})^\pm$ in λ**H(Q)**.

Proof. As ever, by induction on the [$\bot$,$\rightarrow$]-fragment of $\vdash_0$[**CQ**]. □

The $(\dots)^{\pm}$-mappings verify only a *local* version of the usual β-rule $[\beta \to \lambda]$, (*call-by-value*), where, in a *détour* of the form $(\lambda x{:}A.b[\![x]\!])(a)$, the "argument" a must be a p-variable or a λ-abstract (in computer-science jargon, a "value"). So, e.g., $(\lambda x{:}A.b[\![x]\!])(f(c))$, and $b[\![x{:=}f(c)]\!]$ are to be viewed as standing – in general – for distinct proof-objects, under this "interpretation".[28]

As in the case of $(\dots)^{K}$, verified under $(\dots)^{\pm}$ are also $[\eta \to \gamma]$, $[\oint_0 \gamma]$ ("weak diagonalization"), and $[\beta\gamma_0\bot]$. Further, $(\dots)^{+}$ verifies $[\beta\gamma \to]$, while $(\dots)^{-}$ does it *locally* alone [i.e., the "argument" of the γ-abstract $\gamma z{:}\neg(A{\to}B).e$ must be a "value"].

As an idiosyncrasy, in view of the weak/truncated β-behavior, $(\dots)^{-}$ verifies the following curious property:

$$[\gamma\beta \to] \quad \Gamma \vdash f(\gamma z{:}\neg A.e[\![z]\!]) = \gamma y{:}\neg B.e[\![z{:=}\lambda x{:}A.\omega_{\bot}(y(f(x)))]\!]\ [:\ B],$$
$$\text{if } \Gamma[z{:}\neg A] \vdash e[\![z]\!] : \bot \text{ and } \Gamma \vdash f : A \to B,$$

whereas $(\dots)^{+}$ does it, too, but just *locally*, for f restricted to p-variables and λ-abstracts.[29]Notably, $[\gamma\beta \to]$ does not hold – not even in the *local* sense – in other $\lambda\gamma[\mathbf{CQ}]$-theories of concern here: it actually leads to inconsistency (or just "proof-irrelevance") in presence of $[\beta \to \lambda]$ (and $[\eta \to \lambda]$, $[\eta \to \gamma]$). As pointed out in the previous note, one can use the general form of $[\gamma\beta \to]$ – along with other ones of the kind – as a "computing rule", intended to enable us retrieving the normal proofs of a given classical theorem (although "normal" must be taken – in this case – as being relative to a more comprehensive reduction system).

The reader will establish the fact that – like in the case of $(\dots)^{K}$ – the $\to$-extensionality property $[\eta \to \lambda]$, $[\beta\gamma\bot]$, and the "full diagonalization" ($[\oint \gamma]$) are *not* "interpreted" under these translations.

Nearly nothing is known, at the time of writing, about would-be *relative completeness*-properties of the resulting $\lambda\gamma$-formalisms.[30]

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[28] This is, likely, in full agreement with the way some/most of our computing engines do actually operate, but is a rather silly assumption about classical [or other kind of] proofs. Of course, *proving* as actually performed by humans (and machines) *does* depend on the "order of evaluating" proof-components; it is so, in particular, if proofs are "recognized" only by being "seen" first. But we also think that this way of viewing things belongs to psychology (or to "natural history", for that matter), rather than to the *very province of logic*.

[29] The local version – by the way – has a plausible computational interpretation in terms of imperative computer-programs.

[30] To put a name on the outcome, $(\dots)^{+}$ yields the equational system of Felleisen's "calculus of non-local control" (the evaluation of "applications" is done here – "naturally", in some sense: as in Western cultures and present-day computers – *from left to right*). By the end of December 1991, Matthias Felleisen has confirmed the fact that he has partial evidence for the completeness of the equational system induced by $(\dots)^{+}$, as listed – at least implicitly – in the above. The system induced by $(\dots)^{-}$ looks "artificial" (but, likely, only as opposed to the artificial sense of the word "natural", a few lines before). The *call-by-value* calculi induced by $(\dots)^{\pm}$ are *not* proof-theories, in the sense of the present notes. [To our knowledge, neither Felleisen nor his collaborators have ever claimed any proof-theoretic relevance for such formalisms, anyway.] Whether they pertain to "proof-theory" in some other sense of the word, we don't know.

Further reading

A guide to **BHK** (*cca 1925–1993*). The main text is rather parcimonious in references and many topics, supposed familiar, have not been documented bibliographically *ad locum*. What follows is an attempt to remedy this defect by proposing a convenient grouping of the main themes discussed – or only alluded to – in the above and by supplying a *selection of basic references* for each rubric.[31]

A. Lambda-calculus, combinatory logic and type-theory.

- general: [Church 41], [Curry *et al.* 58,72], [Stenlund 72], [Barendregt 84^2], [Hindley *&* Seldin 86], [Seldin 87], [Girard *et al.* 89], [Krivine 90],
- special topics: "completeness" (for $\beta\eta$-normal forms) [Böhm 68], "surjective pairing" and strong normalization [Tait 67], [Stenlund 72], [Barendregt 74], [van Daalen 80], [Troelstra 86], [de Vrijer 87].

B. Intuitionism in general.

- the Brouwer-Heyting bibliography: [van Stigt 90], **Chapter I**, pp. 1–19 (Brouwer), [Niekus *et al.* 81] (Heyting),
- Brouwer texts [logic, philosophy of mathematics]: [Brouwer 07], [Heyting (ed.) 75^R] [van Dalen (ed.) 81,81a,84,92], [van Stigt 90 (ed.)], **Appendices**, pp. 387-505.
- Brouwer's views: [van Dalen 78,80,81b,84,87,90,91], [van Stigt 90],
- systematic expositions: [Heyting 34,55,56,80^R], [Troelstra 69,73], [van Dalen 73], [Dummett 77], [Veldman 87] (based on lectures J. J. de Iongh, Nijmegen), [Troelstra *&* van Dalen 88].

C. Bishop's Constructive Mathematics (BCM).

- [Bishop 67], [Bishop *&* Bridges 85], [Beeson 85].

D. The Automath-family.

- [de Bruijn 80,90] (survey, retrospect, references), [Zucker 77] (AUT-Π, 1975), [Jutting 79 (1977)], [van Daalen 80] (main treatise), [Rezuş 83 (1982)], [Barendregt *&* Rezuş 83], [Rezuş 87 (1983-1986)].

E. The "Heyting logic" (HQ).

- sources: [Kolmogorov 25,32], [Glivenko 28,29], [Heyting 30,34,56],
- history: [Thiel 88] ("Brouwer's logic" before [Heyting 30], references), [Troelstra 78,81,83] (the origins of **HQ**),
- technical aspects: [Troelstra 73], [Zucker 74], [Dummett 75,77], [Pottinger 76,77], [van Dalen 79,86], [Troelstra *&* van Dalen 88],
- the *Minimalkalkül*: [Johansson 36], [Curry 63], [Prawitz 65], [Prawitz *&* Malmnäs 68], [Auge 89],
- the Heyting calculus, proof-equality (in **HQ**, etc.): [Prawitz *passim*], [Kreisel 71], [Stenlund 72], [Troelstra 73,75], [Pottinger 76,77], [Feferman 79], [Cellucci 80], [Girard *et al.* 89].

F. The BHK-interpretation.

- [Kolmogorov 25,32], [Heyting 34,55,80^R], [Freudenthal 37],
- [Kreisel 62,65] (formalization; cf. also [Goodman 70]), [Dummett 77], [van Dalen 79], [Diller 80], [Cellucci 81], [Troelstra 81,83], [Sundholm 83] (references), [Diller *&* Troelstra 84 (1982)], [Martin-Löf 84,85,85a], [Troelstra *&* van Dalen 88] (references).

[31] The bulk of the material on what is traditionally called "proof-theory" can be easily retrieved, from general bibliographies [as, e.g., that of the Ω-group], whereas the main concern of these notes – the *equational theory of classical proofs* – is practically inexistent in the literature. Whence, the "guide" following below, would rather look like a collection of hints for a paleographer: in particular, if he disagrees with the present attempt to a reconstruction, the reader will have to imagine and reshape a "prototype" himself. It is, however, unlikely that there are too many distinct ways of playing the *same* game – i.e., without changing the subject of discussion, once more, by proposing a new logic.

G. "Propositions-as-types" and "condensed detachment".

- [Curry *et al.* 58] (± 1930, according to J. P. Seldin), C. A. Meredith (1950-1952),[32]
- [Läuchli 70 (1965)], [Scott 70 (1968)] [Howard 80 (1969)], [de Bruijn 80 (1970)], [Martin-Löf 71,72 (1970)], [Girard *et al.* 89 (± 1971, Ph D 1972)].
- Meredith "condensed detachment": [Meyer & Bunder 88] (from R. K. Meyer & L. H. Powers 1974), [Kalman 83 (1974)], [Rezuş 82 (1979-1980)], [Hindley & Meredith 90 (1988)].

H. Martin-Löf's Constructive Type Theory (CTT).

- [Martin-Löf 71,72,75,75a,82,84], [Beeson 85], [Rezuş 86,87], [Troelstra & van Dalen 88].

I. "Natural deduction" displays and "sequent"-systems.

- sources: [Jaśkowski 34 (1926)] (presented at the first Polish Mathematical Congress: Lwów 1927), [Gentzen 35 (1932-1934)] (Ph D: Göttingen 1934, using previous work of Paul Hertz: Göttingen 1922-1929),
- "nested blocks" ("sub-proof" style): [Fitch 52 (1950)], [de Bruijn 78,80,81] (using – since 1968 – "abstraction-displays", in *Automath* and the *Mathematical Vernacular* [Dutch: *WOT*], based – apparently – on the "flag"-notation of Hans Freudenthal; cf. also [Nederpelt 77,87], for further developments on *WOT*, and [de Bruijn 90], for historical details),
- ground-work (fresh start): [Prawitz 65 (1964)], [Prawitz 71,73,81a]
- surveys: [Tennant 78], [Sundholm 83a], (cf. also [Auge 89]),
- generalizations: [Schröder-Heister 81,82,82,84,84a,85], [Belnap 82], [Gabbay 90,91], [de Queiroz & Gabbay 91,92,92a].

J. "Beweistheorie" *vs* "general proof-theory".

- [Hilbert & Bernays 31,34], [Gentzen 35 (1932-1934)], [Kreisel 71], [Prawitz 73,74,81].

K. "Meaning theory" (for [the HQ-] proof-operations).

- [Prawitz 77,79], [Cellucci 80,81], [Schröder-Heister 81,82,83,84,84a,85], [Martin-Löf 85,85a], [Sundholm 83,86], [Tieszen 89] (a phenomenological [Husserl] interpretation).

L. "Negative" translations (at a provability-level).

- [Kolmogorov 25], [Glivenko 28,29] (main remark), [Gödel 33], [Gentzen 74 (1933)], [Kuroda 51], [Łukasiewicz 52], [Cellucci 69], [Löb 76], [Friedman 78], [Leivant 85 (1981)].

M. The "Law of Clavius" and the DQ-logic.

- the "Law of Clavius": [Euclid **Elementa** IX.12], [Cardano 1663 (1570)], [Clavius 1611 (1574)], [Saccheri 1697,1733]; secondary literature: [Vailati 03,03a,04,04a], [Kneale 57], [Kneale & Kneale 62^R], [Miralbell 87], [Nuchelmans 92] (further references),
- **DQ**-logic (Curry's *L***D**): [Curry 52 (1950)], [Curry 57^2,63], [Seldin 89].

N. Classical proofs, proof-semantics.

- [Prawitz & Malmnäs 68], [Prawitz 81a (1975)], [Helman 83,87] (likely earlier: ± 1978, following N. D. Belnap Jr.), [Rezuş 81,87a,88,89,90,91,93]; other (recent) approaches: [Mondadori 88], [Bunder 90], [Girard 91].

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[32] Carew A. Meredith (1904–1976), a former student of Jan Łukasiewicz in Dublin, has formulated independently a combinatory variant of the "propositions-as-types" isomorphism, relying – apparently – on considerations derived from a proof-method that was popular in the early Polish school, due to J. Łukasiewicz and A. Tarski ± 1925 [*sic*!]. The original "Polish method" concerned likely the "protothetics" of Stanisław Leśniewski. It has never been recorded in print, and is lost now [no survivors; information confirmed by David Meredith: December 1979]. Cf. [Prior 62^2] *Appendix II*, [Meredith & Prior 63], [Meredith 77], [Rezuş 82 (1979-1980)], [Hindley & Meredith 90 (1988)]. The C. A. M. bibliography can be found in [Meredith 77].

References
and
a Bibliographical Guide to BHK*

Auge, Karlheinz

1989 **Zur linearen Strukturierung des natürlichen prädikatenlogischen Schließens in der positiven Minimallogik, der Minimallogik, der intuitionistischen und der klassischen Logik**, [Ph D Diss. = *Inaugural-Dissertation*, *Ruhr-Universität* Bochum 1989], Bochum 1989, 115 pp. △

Barendregt, Hendrik Pieter

1974 *Pairing without conventional restraints*, **Zeitschrift für mathematische Logik und Grundlagen der Mathematik 20**, 1974, pp. 289–306.

1984^2 **The Lambda Calculus, Its Syntax and Semantics**, North Holland, Amsterdam, etc. 1984^2. △

Barendregt, Henk [Hendrik Pieter] and Adrian Rezuş

1983 *Semantics for Classical Automath and related systems*, **Information and Control 59**, 1983, pp. 127–147. [Rev. **MR 86g**:03100 and **Zbl 564**.68060.] △

Beeson, Michael

1985 **Foundations of Constructive Mathematics – Metamathematical Studies**, Springer Verlag, Berlin, etc. 1985 [*Ergebnisse der Mathematik und ihre Grenzgebiete 3.6*].

Belnap, Nuel D.

1982 *Display logic*, **Journal of Philosophical Logic 11**, 1982, pp. 375–417.

Bishop, Errett W.

1967 **Foundations of Constructive Analysis**, McGraw Hill, New York NY 1967.

Bishop, Errett W. and Douglas Bridges

1985 **Constructive Analysis**, Springer Verlag, Berlin, etc. 1985 [*Grundlehren der mathematischen Wissenschaften 279*]. (A revised version of [Bishop 67].)

Böhm, Corrado

1968 *Alcune proprietà delle forme β-η-normali nel λ-***K***-calcolo*, **Pubblicazioni del Istituto per le Applicazioni del Calcolo** [CNR, Rome] **696**, 1968, pp. 1–19 [*separatum*].

Bridges, Douglas

1985 *see* [Bishop & Bridges 85].

Brouwer, Luitzen Egbertus Jan

1907 **Over de grondslagen der wiskunde**, [Ph D Diss., University of Amsterdam 1907], Maas & van Suchtelen, Amsteradm/Leipzig 1907, 183 pp. (Reprinted in: [van Dalen 81^R]. Translation into English [*On the Foundations of Mathematics*, by A. Heyting], in: [Brouwer 75^R], vol. **1**, pp. 11–97, with [*Stellingen/Statements*], pp. 98–101.) △

1975^R *see* [Heyting 75^R]. △

*Items marked △ in the bibliography are not cited explicitly in the main text; they are intended to document – rather *selectively* – **either** *background issues* and *historical aspects* of the main theme **or** *assorted topics*, and matters *referred to obliquely* in the above. The list has been slightly updated on revision, in July 1993, and in January 2006. Items marked ∇ include last minute additions, and, mainly, web-references. We are grateful to all those authors – too many to be mentioned individually here – who have kindly provided along the years bibliographical pointers to their colleagues' work, as well as to their own. — Worth mentioning separately among the recent updates are (1^o) the marvellous biography of L. E. J. Brouwer, due to the indefatigable Dirk van Dalen [van Dalen 2001], teacher and friend, (2^o) the online *Automath Archive*, stored at the Eindhoven University of Technology (2004), a tribute to Nicolaas G. de Bruijn – our last teacher in maths & meta –, by his many students and co-workers [a huge amount of work, indeed], and, last but not least, (3^o) the Elsevier monograph, **Lectures on the Curry-Howard Isomorphism** – on the very subject of these notes –, due to Morten Heine B. Sørensen [Copenhagen] and Peter Urzyczyn [Warsaw], scheduled in print for 2006; originally a one-semenster graduate / Ph D course held at DIKU [Computer Science Department, University of Copenhegen], Copenhagen, 1998–1999.

1981 *see* [van Dalen 81a]. △
1984 *see* [van Dalen 84]. △
1992 *see* [van Dalen 92]. △
de Bruijn, Nicolaas G.
1978 **Taal en structuur van de wiskunde** [*The Language and Structure of Mathematics*], Lecture Notes, University of Eindhoven, Department of Mathematics and Computing Science, Spring Semester, 1978. (See **Euclides 55**, 1979/1980, pp. 7–12, 66–72, 262–268, 429–435, and [Nederpelt 87].) △
1980 *A survey of the Automath project*, in: J. P. Seldin and J. R. Hindley (eds.) **To H. B. Curry**, **Essays on Combinatory Logic, Lambda Calculus and Formalism**, Academic Press, London, etc. 1980, pp. 579–606. △
1981 *Formalizing the Mathematical Vernacular*, University of Eindhoven, Department of Mathematics and Computing Science, March 1981, 48 pp. △
1990 *Gedachten rondom Automath* [*Reflections on Automath*], University of Eindhoven, Department of Mathematics and Computing Science, March 1990, 24 pp. (For an English version of this report see also Rob Nederpelt, J. H. Geuvers, and Roel C. de Vrijer (eds.) *Selected Papers on Automath*, North-Holland, Amsterdam, etc. 1994.) △ ∇
Bunder, Martin W.
1988 *see* [Meyer *&* Bunder 88]. △
1990 *Combinatory logic and lambda calculus with classical types*, University of Wollongong, Department of Mathematics, Wollongong NSW (Australia), [typescript], 1990-1991 (?), 11 pp. △
1991 *see* [Meyer *et al*. 91]. △
Cardano, Gerolamo (1501–1576)
1663 **Opera omnia** [*Hyeronymi Cardani Mediolanensis*], edited by Karl Spon, 10 vols. in f°, Lugduni [i.e., Lyons, France], 1663. (Reprographic reprint: Fromann Verlag, Stuttgart/Bad Cannstatt and Johnson Reprint Corporation New York/London 1966/1967. The *Opus novum De proportionibus* – first issued in Basle 1570 – is printed in vol. **4**. Cf. pp. 579–581. The original edition [1570] is now available online at the *Archimedes Project* [cf. `http://archimedes.mpiwg-berlin.mpg.de/cgi-bin/toc/toc.cgi` = author index]; see also `http://equivalences.org`, sub *équivalences/éditions*, 2006.) ∇
Cellucci, Carlo
1969 *Un'osservazione sul teorema di Minc-Orevkov*, **Bolletino dell'Unione Matematica Italiana** (4), **2**, 1969, pp. 1–8. (The G. E. Minc *&* V. P. Orevkov paper is in: **Soviet Mathematics - Doklady 4**, 1963, pp. 1365–1366 [= **Doklady Akademij Nauk SSSR 152**, 1963, pp. 553–554].) △
1980 *Proof theory and theory of meaning*, in: M. L. dalla Chiara (ed.) **Italian Studies in the Philosophy of Science**, D. Reidel, Dordrecht, 1980, pp. 13–29. △
1981 *A calculus of constructions as a representation of intuitionistic logical proofs*, in: S. Bernini (ed.) **Atti del Congresso nazionale di logica**, Montecatini Terme, Italy, October 1–5, 1979, Bibliopolis, Naples 1981, pp. 175–193. △
Church, Alonzo
1936 **Mathematical Logic**, Lecture Notes [by F. A. Ficken, H. G. Landau, H. Ruja, R. R. Singleton, N. E. Steenrod, J. H. Sweer, and F. J. Weyl], Princeton University, October 1935 – January 1936 [typescript], Princeton NJ 1936, 113 pp. (Description based on a copy owned by H. B. Curry, located by J. P. Seldin [1979] in Curry's personal library. See [Rezuş 81] for details, and, possibly, http://www.equivalences.org, sub *équivalences/éditions*, for a textual record.) △ ∇
1941 **The Calculi of Lambda-conversion**, Princeton UP, Princeton NJ [*Annals of Mathematics Studies 6*]. (Second printing: 1951^2, reprint: University Microfilms, Ann Arbor Michigan, 1963^R. This monograph is a revision of the author's Princeton Lectures [Church 36].)

Clavius, Christophorus [= Klau, Christoph], SJ (1537–1612)

1611 [*Christophori Clavii Bambergensis e Societate Jesu*] **Opera mathematica** [*V tomis distributa ab auctore nunc denuo correcta et plurimis locis aucta*...], 5 vols., in f°, edited by Reinhard Eltz and published by Anton Hierat, Moguntiae [i.e., Mainz] 1611–[1612]. (Microfilm-edition, issued by the Pius XII Memorial Library, St. Louis University [not seen, cf. the **NUC**-entries]. Copy consulted in "The L .E. J. Brouwer Collection", *Provinciale Bibliotheek van Friesland*, Leeuwarden [formerly, Brouwer's personal library]. Vol. **1**: *Commentaria In Euclidis Elementa geometrica* [*et*] *In Sphaerica Theodosii*..., in two parts: **1.1**, pp. 1–638, and **1.2**, pp. 1–248, with separate page-numbering. Cf. also http://www.equivalences.org, sub *équivalences/éditions*.) ▽

Craig, William

1958 *see* [Curry *et al*. 58]. △

Curry, Haskell Brooks

1952 *The system L***D**, **Journal of Symbolic Logic 17**, 1952, pp. 35–42. (Contains results reported at the *International Congress of Mathematicians*, Cambridge Mass., in 1950. The first note, appearing in Curry's card index, and referring to [the "law" of] Clavius is in a reference to/from [Łukasiewicz 52], dated September 29, 1953 [information verified, in the Curry archives, by J .P. Seldin: correspondence, 1992]. Apparently, H. B. Curry ignored, as of 1950, the ramified XVI–XVII century debates on the *consequentia mirabilis*; his work on *L***D** is – more likely – technically and/or philosophically motivated, rather than by reference to *historical* antecedents.)

1957^2 **A Theory of Formal Deducibility**, Notre Dame UP, Notre Dame IN 1957^2 [*Notre Dame Mathematical Lectures 6*] (1950^1). △

1963 **Foundations of Mathematical Logic**, McGraw Hill, New York NY 1963. (Reprints: Dover Publications Inc., New York NY 1977^R, 1984^R.)

Curry, Haskell Brooks *et al*.

Combinatory Logic, (2 vols.), North Holland, Amsterdam, etc. 1958, 1972. (Vol. **1**, with Robert Feys and William Craig; vol. **2**, with Jonathan P. Seldin and J. Roger Hindley. Several reprints.) △

van Daalen, Diederik Ton

1980 **The Language Theory of Automath**, [Ph D Diss., University of Eindhoven 1980], Wibro, Helmond 1980, 307 pp. (Cf. also http://alexandria.tue.nl/extra3/proefschrift/PRF3B/8001697.pdf.) ▽

van Dalen, Dirk

1973 *Lectures on intuitionism*, in: A. R. D. Matthias and H. R. Rogers (eds.) **Cambridge Summer School in Mathematical Logic 1971**, Springer Verlag, Berlin, etc. 1973 pp. 1–94. [*Lecture Notes in Mathematics 337*]. △

1978 *Brouwer: the genesis of his intuitionism*, **Dialectica 32**, 1978, pp. 291–303. △

1979 *Interpreting intuitionistic logic*, in: P. C. Baayen, D. van Dulst, and J. Oosterhoff (eds.) **Proceedings <of the> Bicentennial Congress <of the>** *Wiskundig Genootschap*, Part **1**, Mathematisch Centrum [*CWI*], Amsterdam 1979, pp. 133–148 [*MC Tracts 100*]. △

1980 *L. E. J. Brouwer en de eenzaamheid van het gelijk*: *Biografie*,..., [*L. E. J. B. and the solitude of being-right, A biography*, etc.], in: **Vrij Nederland 2**, February 21, 1981, [*Bijlage*], pp. 4–23. (Special issue, published with the occasion of the 100th anniversary of Brouwer's birthday [February 27, 1881], contributed by D. v. D. [this biography], Hans Freudenthal [topology], pp. 26–27, and Gerrit Krol [intuitionism], pp. 28–34. Reprinted, in a slightly revised form, in [van Dalen 84], pp. 135–170.) △

1986 *Intuitionistic logic*, in: D. Gabbay and F. Guenthner (eds.) **Handbook of Philosophical Logic 3 Alternatives to Classical Logic**, D. Reidel [Kluwer Academic Publishers], Dordrecht, etc. 1986, pp. 225–239. (A slightly revised version of the **Preprint 209**, September 1981, University of Utrecht, Department of Mathematics.) △

1987 *The war of the frogs and the mice, or the crisis of the Mathematische Annalen*, University of Utrecht, Department of Philosophy, **Logic Group Preprint Series 28**, December 1987, 32 pp. (Cf. also **Math. Intelligencer 12**, 1990, pp. 17–31.) △ ∇

1988 *see* [Troelstra *&* van Dalen 88]. △

1990 *L. E. J. Brouwer, wiskundige en mysticus*, [*L. E. J. B., the mathematician and the mystic*], University of Utrecht, Department of Philosophy, **Logic Group Preprint Series 54**, June 1990, 31 pp. △

1991 *Brouwer's dogma of languageless mathematics and its rôle in his writings*, in: E. Heijerman *&* H. W. Schmitz (eds.) **Significs, Mathematics and Semiotics. The Signific Movement in the Netherlands.** *Proceedings of the International Conference: Bonn, 19–21 November 1976*, Nodus Publikationen, Münster 1991, pp. 33–43. △

2000 *The development of Brouwer's intuitionism*, in: K. F. Jørgensen, V. Hendricks, S. A. Perdersen (eds.) **Proof Theory – History and Philosphical Significance**, Kluwer, Dordrecht 2000, pp. 117–152 [*Synthese Library 292*]. △ ∇

2001 **L. E. J. Brouwer – Een biografie** [*L. E. J. B., A Biography*], Uitegeverij Bert Bakker, Amsterdam, 2001, 2002^2. (See also: **Mystic, Geometer and Intuitionist: The Life of L. E. J. Brouwer**, *Part I. The Dawning Revolution*, Oxford UP, Oxford 1999.) △ ∇

van Dalen, Dirk (ed.)

1981 **L. E. J. Brouwer: Over de grondslagen der wiskunde**, *aangevuld met ongepubliceerde fragmenten, correspondentie met D. J. Korteweg, recensies door G. Mannoury, etc. Inleiding door D. van Dalen*, Mathematisch Centrum [*CWI*], Amsterdam 1981 [*MC Varia 1*]. (A reprint of [Brouwer 07] with additions: unpublished fragments and other documents, edited, with an introduction, by Dirk van Dalen, on the occasion of the 100th anniversary of Brouwer's birthday, under the auspices of the *Wiskunding Genootschap*, Amsterdam, and the *Brouwer-Archief*. Cf. also **L. E. J. Brouwer en De Grondslagen der Wiskunde** [*L. E. J. Brouwer and The Foundations of Mathematics*], Epsilon, Utrecht 2001.) △ ∇

1981a **Brouwer's Cambridge Lectures on Intuitionism**, Cambridge UP, Cambridge [UK] 1981. △

1984 **Carel Steven Adama van Scheltema *&* Luitzen Egbertus Jan Brouwer: Droeve Snaar, Vriend van Mij, Brieven** [*<You> Sorrow <Violin> String, My Friend, Letters*], De Arbeiderspers, Amsterdam 1984. (Annotated correspondence, with an introduction, and a biographical sketch by D. v. D.) △

1992 **L. E. J. Brouwer: Intuitionismus**, *herausgegeben von Dirk van Dalen*, B. I.–Wissenschaftsverlag, Mannheim, etc. 1992, 165 pp. (appearead: August 1992 [ISBN 3-411-15371-71]). △

Diller, Justus

1980 *Modified realizability and the formulae-as-types notion*, in: J. P. Seldin and J. R. Hindley (eds.) **To H. B. Curry, Essays on Combinatory Logic, Lambda Calculus and Formalism**, Academic Press, London, etc. 1980, pp. 491–502. △

Diller, Justus and Anne S. Troelstra

1984 *Realizability and intuitionistic logic*, **Synthese 60**, 1984, pp. 253–282. (A revised version of the **Report 82-12**, 1982, University of Amsterdam, Mathematical Institute.) △

Dummett, Michael A. E.

1975 *The philosophical basis of intuitionism*, in: H. E. Rose *et al.* (eds.) **Logic Colloquium '73**, North Holland, Amsterdam, etc. 1975, pp. 5–40. △

1977 **Elements of Intuitionism**, (With the assistance of Roberto Minio), Clarendon Press, Oxford UK 1977 (1978^R) [*Oxford Logic Guides*]. △

Euclid

Elementa, in: [*Euclidis*] **Opera omnia 1–4**, edited by J. L. Heiberg and H. Menge, 9 vols. Teubner Verlag, Leipzig 1883–1885 [–1916]. (The Heiberg text **1–4**, has been revised by E. S. Stamatis, Teubner Verlag, Leipzig, **1–4**, 1969–1974. Latin paraphrasis, with a commentary by Clavius, in: [Clavius 1611], vol. **1.1**. Translation into English, with a commentary, by Sir Thomas Little Heath: **The Thirteen Books of Euclid's Elements**, Cambridge UP, Cambridge [UK] 1926². Reprint: Dover Publ. Inc., New York NY, 3 vols. 1956^R.)

Feferman, Solomon

1979 *What does logic have to tell us about mathematical proofs?*, **Mathematical Intelligencer 2**, 1979, pp. 20–24. △

Feys, Robert

1958 *see* [Curry *et al*. 58]. △

Felleisen, Matthias

1987 **The Calculi of λ_v-CS Conversion, A Syntactic Theory of Control and State in Imperative Higher Order Programming Languages**, [Ph D Diss., Indiana University, Bloomington IN 1987], Indiana University, Department of Computer Science, Bloomington IN, **Technical Report 226**, August 1987, xvii + 230 pp. (See also: *A syntactic theory of sequential control*, [with Daniel P. Friedman, Eugene Kohlbecker, and Bruce Duba], in: **Theoretical Computer Science 52**, 1987, pp. 205–237, and *The revised report on the syntactic theories of sequential control and state*, [with Robert Hieb], Rice University, Department of Computer Science, Houston TX, **Technical Report 89-100**, December 1989, 30 pp.) △

Fitch, Frederick Brenton

1952 **Symbolic Logic, An Introduction**, Ronald Press, New York NY 1952. △

Friedman, Harvey

1978 *Classically and intuitionistically provable recursive functions*, in: D. S. Scott and G. H. Mueller (eds.) **Higher Set Theory**, Springer Verlag, Berlin, etc. 1978, pp. 21–28 [*Lecture Notes in Mathematics 699*]. △

Freudenthal, Hans

1937 *Zur intuitionistischen Deutung logischer Formeln*, **Compositio Mathematica 4**, 1937, pp. 112–116. (Received December 27, 1934. Reprinted in: [Heyting 80^R], pp. 331–335, followed by the *Remarks* of Heyting, dated October 16, 1935, and a last word [*Nachwort*] by the author [six lines], dated April, 29, 1936 [= **Compositio Mathematica 4**, 1937, pp. 117–118, reprinted in: *ibid*., pp. 336–337]. The issue containing the paper and the exchange with Heyting has appeared in 1936. For a previous [1930] debate on the subject, as reflected in the Freudenthal-Heyting correspondence, see [Troelstra 83], pp. 203–208.) △

Gabbay, Dov M.

1990 *Labelled deductive systems – a position paper*, in: **Logic Colloquium '90**, Springer Verlag, Berlin, etc. [to appear]. △

1991 *LDS – labelled deductive systems*, **CIS–Bericht 90–92**, [*Centrum für Informations- und Sprachverarbeitung*], University of Munich, February 1991, 265 pp. (Revised many times since into "partial working drafts" of a book, intended for Oxford University Press. This material is also available "in the form of a technical report" [**Labelled Deductive Systems, Part I**, Imperial College, Department of Computing, London SW7 2BZ, "1st draft September 1989, 7th expanded draft April 1992"].) △

1991a *see* [Gabbay *&* de Queiroz 91]. △

1992 *see* [Gabbay *&* de Queiroz 92]. △

1992a *see* [Gabbay *&* de Queiroz 92a]. △

Gabbay, Dov M. and Ruy J. G. B. de Queiroz

1991 *The functional interpretation of the existential quantifier*, Imperial College, Department of Computing, London SW7 2BZ, UK, December 16, 1991, 41 pp. △

1992 *Extending the Curry-Howard interpretation to linear, relevant, and other resource logics*, **Journal of Symbolic Logic 57**, 1992, pp. 1319–1365. △

1992a *An introduction to labelled natural deduction*, Imperial College, Department of Computing, London SW7 2BZ, UK, September 18, 1992, 80 pp. (Lecture Notes for the *Third Advanced Summer School in Artificial Intelligence*, Azores, September 21–25, 1992 and "draft proposal for a book" [**The Functional Interpretation of Non-Classical Logics**]. △

Gentzen, Gerhard

1935 *Untersuchungen über das logische Schliessen* [Ph D Diss., University of Göttingen 1934], **Mathematische Zeitschrift 39**, 1935, pp. 176–210, and 405–431. (Translation [*Investigation into logical deduction*, by M. E. Szabo] in: M. E. Szabo (ed.) **The Collected Papers of Gerhard Gentzen**, North Holland, Amsterdam, etc. 1969, pp. 68–131.)

1974 *Über das Verhältnis zwischen intuitionistischer und klassischer Arithmetik*, **Archiv für mathematische Logik 16**, 1974, pp. 119–132. (Submitted to **Mathematischen Annalen** [received March 15, 1933], then withdrawn. Translation [*On the relation between intuitionist and classical arithmetic*, by M. E. Szabo] in the Szabo-collection, pp. 53–67.)

Girard, Jean-Yves

1991 *A new constructive logic: classical logic*, INRIA-Rocquencourt, France, **Rapports de Recherche [Programme 2]** N^o 1443, June 1991, 51 pp. (Published in: **Mathematical Structures in Computer Science 1**, 1991, pp. 255–296. See also *On the unity of logic*, *ibid.* N^o 1467, June 1991, 18 pp. [to appear] and the author's contributions to `types.theory.lcs.mit.edu`, on *Polarities, behaviours and related topics* [November 29, 1991], *On the sex of angels* [December 6, 1991], etc.) △

Girard, Jean-Yves *et al.* [Paul Taylor and Yves Lafont]

1989 **Proofs and Types**, Cambridge UP, Cambridge [UK] 1989 [*Cambridge Tracts in Theoretical Computer Science 7*]. (Lecture Notes for a *Cours DEA 86–87*, University of Paris VII. Revised, translated into English, and contributed with 2 Appendices, by Paul Taylor and Yves Lafont. For the main author's early [1986–1990] views on proofs in classical logic [as recorded by Yves Lafont], see the *Appendix B.*, especially its introductory section *B.1 Classical logic is not constructive* [pp. 150–151]. Cf. [Girard 91]. The Cambridge Tract has been issued as a reprint 1990^R, with corrections. Cf. also http://www.cs.man.ac.uk/ pt/stable/prot.pdf.) △ ▽

Glivenko, V. I.

1928 *Sur la logique de M. Brouwer*, **Académie Royale de Belgique, Bulletin de la Classe des Sciences** (5) **14**, 1928, pp. 225–228.

1929 *Sur quelques points de la logique de M. Brouwer*, **Académie Royale de Belgique, Bulletin de la Classe des Sciences** (5) **15**, 1929, pp. 183–188.

Gödel, Kurt

1933 *Zur intuitionistischer Arithmetik und Zahlentheorie*, in: **Ergebnisse eines mathematischen Kolloquiums 4**, 1933, pp. 34–38. (Reprinted in: S. Feferman *et al.* (eds.) **Kurt Gödel: Collected Works**, vol. **1**, Oxford UP, Oxford UK, etc. 1986, pp. 286–295.)

Goodman, Nicolas

1970 *A theory of constructions equivalent to arithmetic*, in: A. Kino *et al.* (eds.) **Intuitionism and Proof Theory**, North Holland, Amsterdam, etc. 1970, pp. 101–120. (Cf. also: *The arithmetic theory of constructions*, in: A. R. D. Matthias and R. H. Rogers (eds.) **Cambridge Summer School in Mathematical Logic 1971**, Springer Verlag, Berlin, etc. 1973, pp. 278–298 [*Lecture Notes in Mathematics 337*] and *The faithfulness of the interpretation of arithmetic in the theory of constructions*, **Journal of Symbolic Logic 38**, 1973, pp. 453–459.) △

Griffin, Timothy G.

1990 *A formulae-as-types notion of control*, in: **Conference Record of the 17th Annual ACM Symposium on Principles of Programming Languages** (San Francisco CA, January 17–19, 1990), ACM Press 1990, pp. 47–58. △

Helman, Glen H.

1977 **Restricted Lambda Abstraction and the Interpretation of some Non-Classical Logics**, [Ph D Diss., University of Pittsburgh 1977], Pittsburgh PA [University Microfilms, Ann Arbor Michigan] 1977, iv + 217 pp. (*Propositions-as-types* for "positive [propositional] relevant logics". The material mentioned under [Helman 83,87] was, apparently, *not* yet available in 1977–1978 [Helman: correspondence, 1978]. See also this author's contributions to N. D. Belnap *et al*. **Entailment.** *The Logic of Relevance and Necessity*, volume **2**, Princeton UP, Princeton NJ 1992.) △

1983 *An interpretation of classical proofs*, **Journal of Philosophical Logic 12**, 1983, pp. 39–71. △

1987 *On the equivalence of proofs involving identity*, **Notre Dame Journal of Formal Logic 28**, 1987, pp. 297–321. △

Heyting, Arend

1930 *Die formalen Regeln der intuitionistischen Logik*, **Sitzungsberichte der Preussischen Akademie von Wissenschaften,** *Phys.-math. Klasse*, 1930, pp. 42–56. (Reprinted in: [Heyting 80^R], pp. 191–205. Excerpts in: [Troelstra 78], pp. 153–162.)

1934 **Mathematische Grundlagenforschung, Intuitionismus, Beweistheorie**, Springer Verlag, Berlin, etc. 1934 (reprographic reprint 1974^R) [*Ergebnisse der Mathematik und ihre Grenzgebiete 3.4*]. △

1955 **Les Fondements des mathématiques. Intuitionnisme. Théorie de la démonstration**, Gauthier Villars, Paris-Louvain 1955. (A translation of [Heyting 34], with additions.) △

1956 **Intuitionism, An Introduction**. North Holland, Amsterdam 1956^1 (revised: 1966^2, and 1971^3). △

1980^R **Collected Papers**, (edited by the) Mathematical Institute, University of Amsterdam: Amsterdam 1980^R, [xvii + 821 pp.] (Collected reprographic reprints.)

Heyting, Arend (ed.)

1975^R **L. E. J. Brouwer: Collected Works**, vol. **1**: *Philosophy and Foundations of Mathematics*, North-Holland/American Elsevier, Amsterdam, etc. 1975. △

Hilbert, David and Paul Bernays

Grundlagen der Mathematik, (2 vols.), Springer Verlag, Berlin 1934, 1939 (reprinted: 1968^2, 1970^2) [*Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 40, 50*]. △

Hindley, J. Roger

1972 *see* [Curry *et al*. 72]. △

1997 **Basic Simple Type Theory**, Cambridge UP, Cambridge [UK] 1997. △ ▽

Hindley, J. Roger and David Meredith

1990 *Principal type schemes and condensed detachment*, **Journal of Symbolic Logic 55**, 1990, pp. 90–105.

Hindley, J. Roger and Jonathan P. Seldin

1986 **Introduction to Combinators and λ-Calculus**, Cambridge UP, Cambridge [UK] 1986 [*London Mathematical Society Student Texts 1*]. △

Howard, William A.

1980 *The Formulae-as-Types notion of construction*, in: J. P. Seldin and J. R. Hindley (eds.) **To H. B. Curry**, **Essays on Combinatory Logic, Lambda Calculus and Formalism**, Academic Press, London, etc. 1980, pp. 479–490. (Notes written in 1969; slightly revised in 1979–1980.) △

Jaśkowski, Stanisław

1934 *On the rules of supposition in formal logic*, **Studia Logica 1**, 1934, pp. 5-32. (Also in: S. McCall (ed.) **Polish Logic 1920–1939**, At the Clarendon Press [= Oxford UP], Oxford UK 1967, pp. 232–258. Contains "results obtained... in 1926".) △

Johansson, Ingebrigt

1936 *Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus*, **Compositio Mathematica 4**, 1936, pp. 119–136. (The issue containing the paper has appeared in 1935.)

[van Benthem] Jutting, Lambertus S.

1979 **Checking Landau's "Grundlagen" in the Automath System**, [Ph D Diss., University of Eindhoven 1977], Mathematisch Centrum [*CWI*] Amsterdam 1979, 120 pp. [*MC Tracts 83*]. (Cf. also http://alexandria.tue.nl/extra1/PRF2B/7701940.pdf.) △

Kalman, John

1983 *Condensed detachment as a rule of inference*, **Studia Logica 42**, 1983, pp. 443–451. (Cf. **JSL 39**, 1974, page 206.) △

Klau, Christoph, SJ (1537–1612)

1611 *see* [Clavius 1611].

Kneale, Martha

1962 *see* [Kneale & Kneale 62^R]. △

Kneale, William

1957 *Aristotle and the consequentia mirabilis*, **Journal of Hellenic Studies 77**, 1957, pp. 62–66. △

Kneale, William and Martha Kneale

1962^R **The Development of Logic**, Clarendon Press, Oxford 1962^R. (On *consequentia mirabilis* and the Cardano-Clavius-Saccheri episodes, see **V.4**.) △

Kolmogorov, Andrei Nikolaevič

1925 *O principe* **tertium non datur** (Russian), **Matematičeskij Sbornik 32**, 1925, pp. 646–667. (Translation [*On the principle of the excluded middle*, by J. van Heijenoort] in: J. van Heijenoort (ed.), **From Frege to Gödel, A Source Book in Mathematical Logic, 1879–1931**, Harvard UP, Cambridge Mass. 1967^1, 1970^R, pp. 414–437.)

1932 *Zur Deutung der intuitionistischen Logik*, **Mathematische Zeitschrift 35**, 1932, pp. 58–65. (Submitted: January 1931.) △

Kreisel, Georg

1962 *Foundations of intuitionistic logic*, in: E. Nagel *et al.* (eds.) **Logic, Methodology and Philosophy of Science I** (1960), Stanford UP, Stanford CA 1962, pp. 192–210. △

1965 *Mathematical logic*, in: T. L. Saaty (ed.) **Lectures on Modern Mathematics**, vol. **3**, John Wiley & Sons Inc., New York NY, etc. 1965, pp. 95–195. (Reprint: 1967^2.) △

1971 *A survey of proof theory II*, in: J. E. Fenstad (ed.) **Proceedings of the Second Scandinavian Logic Symposium**, North Holland, Amsterdam 1971, pp. 109–170. (Part I in: **JSL 33**, 1968, pp. 321–388.) △

Krivine, Jean-Louis

1990 **Lambda-calcul, types et modèles**, Masson, Paris, etc. 1990. △

Kuroda, Sigekatu

1951 *Intuitionistische Untersuchungen der formalistischen Logik*, **Nagoya Mathematical Journal 2**, 1951, pp. 35–47. △

Läuchli, Herbert

1970 *An abstract notion of realizability for which the intuitionistic logic is complete*, in: A. Kino *et al.* (eds.) **Intuitionism and Proof Theory**, North Holland, Amsterdam, etc. 1970, pp. 227–234. (An algebraic version of the *propositions-as-types*, applying to the full Heyting logic. Relies on work done earlier [± 1965]. Cf. **JSL 30**, 1965, page 263.) △

Lafont, Yves

1989 *see* [Girard *et al.* 89]. △

Leivant, Daniel

1985 *Syntactic translations and provably recursive functions*, **Journal of Symbolic Logic 50**, 1985, pp. 682–684. △

Löb, M. H.

1976 *Embedding first-order predicate logic in fragments of intuitionistic logic*, **Journal of Symbolic Logic 41**, 1976, pp. 705–718. △

Łukasiewicz, Jan

1952 *On the intuitionistic theory of deduction*, **Koninklijke Nederlandse Akademie van Wetenschappen**, *Proceedings*, Series A, **55**, 1952, (3) [= **Indagationes mathematicae 14**], pp. 202–212. (Cf. also K. J. Cohen's *Remark* on the paper and J. Ł.'s *Comment* on it, in: *ibid.*, **56**, 1953, pp. 111–112 [= **Indagationes mathematicae 15**].) △

Malmnäs, Per-Erik

1968 *see* [Prawitz *&* Malmnäs 68]. △

Martin-Löf, Per

1971 **A Theory of Types**, University of Stockholm, Mathematical Institute, **Report 71-3** (February 1971, revised: October 1971), 57 pp. △

1972 **An Intuitionistic Theory of Types**, University of Stockholm, Mathematical Institute, Technical Report 1972, 87 pp. △

1975 *An intuitionistic theory of types: predicative part*, in: H. E. Rose and J. C. Sheperdson (eds.) **Logic Colloquium '73**, North Holland, Amsterdam, etc. 1975, pp. 73–118. △

1975a *About models for intuitionistic type theories and the notion of definitional equality*, in: S. Kanger (ed.) **Proceedings of the Third Scandinavian Logic Symposium**, North Holland, Amsterdam, etc. 1975, pp. 81–109. △

1982 *Constructive mathematics and computer programming*, in: L. J. Cohen *et al.* (eds.) **Logic, Methodology and Philosophy of Science VI**, North Holland, Amsterdam, etc. 1982, pp. 153–175. △

1984 **Intuitionistic Type Theory**, Bibliopolis, Naples 1984 [*Studies in Proof Theory*]. (Lectures given in Padova, June 1980. Lecture Notes by Giovanni Sambin.)

1985 *On the meanings of the logical constants and the justifications of the logical laws*, in: **Atti degli incontri di logica matematica**, *Scuola di Specializzazione in Logica Matematica*, *Università degli Studi di Siena*, vol. **2**, 1985, pp. 203–281. (Lectures for the meeting: *Teoria della dimostrazione e filosofia della logica*, University of Siena, April 6-9, 1983; transcription of tape-recordings.) △

1985a *Truth of a proposition, evidence of a judgement, validity of a proof*, Talk for the Workshop *Theories of Meaning*, *Centro Fiorentino di Storia e Filosofia della Scienza*, Florence, June 3-7, 1985, [typescript], 20 pp. △

Meredith, Carew A. and Arthur N. Prior

1963 *Notes on the axiomatics of the propositional calculus*, **Notre Dame Journal of Formal Logic 4**, 1963, pp. 171–187. (One of the rare items – by C. A. Meredith himself – mentioning "condensed detachment" in print. Cf. also [Prior 62^2.) △

Meredith, David

1977 *In Memoriam: Carew Arthur Meredith (1904–1976)*, **Notre Dame Journal of Formal Logic 18**, 1977, pp. 513–516. △

1990 *see* [Hindley *&* Meredith 90]. △

Meyer, Robert K. and Martin W. Bunder

1988 *Condensed detachment and combinators*, The Australian National University, **The Automated Reasoning Project**, Canberra ACT, Report **TR-ARP-8-1988**, September 20, 1988, 78 pp. [to appear]. (An algebraic variant of the *propositions-as-types*, applying to the pure "positive" implication and fragments [the "Fool's Model" of CL], "discovered by Meyer back in 1973-74, building on the approach to the **P**-*W* problem set out by Powers. . . ". Cf. [Powers 76] and [Meyer *et al.* 91].) △

Meyer, Robert K. and Larry [= Lawrence H.] Powers

1976 *The intuitive model for combinatory logic, with application to a conjecture of N. Belnap*, University of Pittsburgh, Department of Philosophy [typescript] 1974-1976 (?), 95 pp. (Written up by R. K. Meyer. Apparently incomplete. Description from Meyer's file at the Australian National University, Research School of Social Sciences, Canberra ACT [Australia]. Documents the original version of the "Fool's Model of Combinatory Logic". See also [Powers 76] and [Meyer & Bunder 88], [Meyer *et al.* 91] for details.) △

Meyer, Robert K. *et al.* [Martin W. Bunder and Lawrence H. Powers]

1991 *Implementing the "Fool's Model" of combinatory logic*, University of Wollongong, Department of Mathematics, Wollongong NSW (Australia), **Preprint 4-91**, March 1991, 23 pp. (Dated: September 14, 1989. Revised version in: **Journal of Automated Reasoning 7**, 1991, pp. 597–630.) △

Miralbell, Ignacio

1987 *La consequentia mirabilis: desarrollo histórico y implicaciones filosóficas*, **Thémata – Revista de filosofia** [Sevilla] **4**, 1987, pp. 79–95. (Cf. **International Philosophical Bibliography – Bibliografisch Repertorium van de Wijsbegeerte** [Louvain] 1990, #2933, page 140.) △

Mondadori, Marco

1988 *Sulla nozione di dimostrazione classica*, **Annali dell'Università di Ferrara** (n.s.), [Section 3: Philosophy], *Discussion Paper* **3**, University of Ferrara 1988. (English summary: *On the notion of a classical proof*, in: C. Cellucci and G. Sambin (eds.) **Temi e prospettive della logica e della filosofia della scienza contemporanee** (Cesena, Italy, 7–10 January 1987), vol. **1**, CLUEB, Bologna 1989. See also: *Classical analytical deduction I, II*, **Annali dell'Università di Ferrara** (n.s.), [Section 3: Philosophy], *Discussion Papers* **1**, 1988, 23 pp., and **5**, 1989, 28 pp.) △

Murthy, Chetan R.

1990 **Extracting Constructive Content from Classical Proofs** [Ph D Diss., Cornell University, Ithaca NY 1990], Cornell University, Department of Computer Science, Ithaca NY, **Technical Report 90-1150**, August 1990, xi + 195 pp. △

Nederpelt [Lazarom], Rob[ert Pieter]

1977 *Presentation of natural deduction*, in: **Recueil de travaux de l'Institut mathématique** [Beograd], (n.s.) **2**, 1977, pp. 115–126. △

1987 **De taal van de wiskunde, Een verkenning van wiskundig taalgebruik en logische redeneerpatronen**, *met een voorwoord van Dirk van Dalen*, [*The Language of Mathematics, An Exploration of the Mathematical Use of Language and the Logical Proof-Patterns, with an Introduction by D. v. D.*] Versluis, Almere 1987. △

Niekus, N. H. *et al.* [H. van Riemsdijk and A. S. Troelstra]

1981 *Bibliography of A. Heyting*, **Nieuw Archief voor Wiskunde** (3), **29**, 1981, pp. 24–35. (Cf. the Heyting Amsterdam-edition 1980, pp. iii–xvii.)

Nuchelmans, Gabriel

1992 *A 17th-century debate on the consequentia mirabilis*, **History and Philosophy of Logic 13**, 1992, pp. 43–58. (See also: **Dilemmatic Arguments. Towards a History of their Logic and Rhetoric**, North Holland, Amsterdam, etc. 1991 [*Verhandelingen der Koninklijke Nederlandse Akademie van Wetenschappen*, *Afdeling Letterkunde* (n.s.), *145*], especially *II.9: Dilemma and consequentia mirabilis*, pp. 115–137.) △

Peckhaus, Volkert

1990 **Hilbertprogramm und kritische Philosophie**, Vandendoeck & Ruprecht, Göttingen 1990 (also issued as a Ph D Diss., Erlangen-Nürenberg Univ. 1990). △ ▽

Peirce, Charles Sanders (1839–1914)

1885 *On the algebra of logic. A contribution to the philosophy of notation*, **American Journal of Mathematics** (1), **7** (no. 2), 1885, pp. 180-202. (Reprinted in: Charles Hartshorne & Paul Weiss (eds.) **Collected Papers of Charles Sanders Peirce**, vol. **3: Exact Logic**: *Published Papers*, The Belknap Press of Harvard UP, Cambridge Mass. 1960^2 [1933^1], **3**.359–**3**.403M. The "fifth icon" [= *"Peirce's Law"*] appears at **3**.384.)

Pottinger, Garrel

1976 *A new way of normalizing intuitionist propositional logic*, **Studia Logica 35**, 1976, pp. 387–408. (Likely the first notice – in print – of the fact that a Gentzen L-system for the Heyting [propositional] logic is a [*special*] form of typed λ-calculus, too. See, however, [Howard 80].) △

1977 *Normalization as a homomorphic image of cut-elimination*, **Annals of Mathematical Logic 12**, 1977, pp. 323–357. △

Powers, Lawrence H.

1976 *On* **P**-W, **Relevance Logic Newsletter 1** (3), 1976, pp. 131–142. (See also the author's *Latest on a logic problem*, in: *ibid.*, pp. 143–172, and [Meyer & Bunder 88], [Meyer *et al*. 91]. Following Robert K. Meyer, L. H. Powers has re-discovered the *propositions-as-types* isomorphism – as well as the Schönfinkel-Curry combinators (!) – during the mid-seventies, while working on the stubborn "**P**-W-problem", posed by Alan Ross Anderson and Nuel Belnap Jr. [cf. **Entailment**, *The Logic of Relevance and Necessity*, volume **1**, Princeton UP, Princeton NJ 1975, **8.11**, pp. 94–95]. Meyer regards this work as a source for what he used to call "the Fool's Model of Combinatory Logic". The original problem – not of direct concern here – was eventually solved by Errol Peter Martin [a former student of Meyer] and Meyer himself, in 1978. Cf. Martin's Ph D Diss. **The P-W-problem**, The Australian National University, Canberra ACT [Australia], July 1978, vi + 117 pp., the Martin-Meyer joint paper *Solution to the* **P**-W*-problem*, in: **JSL 47**, 1982, pp. 869–887, and, possibly, Aleksandar Kron *A constructive proof of a theorem in relevance logic*, in: **ZmLGM 31**, 1985, pp. 423–430.) △

1976a *see* [Meyer & Powers 76]. △

1991 *see* [Meyer *et al*. 91] △

Prawitz, Dag

1965 **Natural Deduction**, **A Proof-Theoretical Study**, [Ph D Diss., University of Stockholm 1965], Almqvist & Wiksell, Uppsala [Stockholm-Gothenburg-Uppsala] 1965 [*Acta Universitatis Stockholmiensis: Stockholm Studies in Philosophy 3*]. ([Pottinger 76], page 404, signalled a – trivial, though fatal – oversight in the [Prawitz 65] definition of (the "positive", λ-) abstraction [= the "discharge function(s)"], equivalent to a "systematic" α-conversion error. This has been repaired, apparently earlier, by the author, in [unpublished] lectures, given in Oxford [1971], yet *without* any use of λ-calculus! Cf. [Prawitz 71].)

1971 *Ideas and results in proof theory*, in: J. E. Fenstad (ed.) **Proceedings of the Second Scandinavian Logic Symposium**, North Holland, Amsterdam, etc. 1971, pp. 235–307. △

1973 *Towards a foundation of general proof theory*, in: P. Suppes *et al*. (eds.) **Logic, Methodology and Philosophy of Science IV**, North Holland, Amsterdam, etc. 1973, pp. 225–250. △

1974 *On the idea of a general proof theory*, **Synthese 27**, 1974, pp. 63–77. △

1977 *Meanings and proofs: on the conflict between classical and intuitionistic logic*, **Theoria** [Lund] **43**, 1977, pp. 2–40. △

1979 *Proofs and the meaning and completeness of the logical constants*, in: J. Hintikka *et al*. (eds.) **Essays on Mathematical and Philosophical Logic**, D. Reidel, Dordrecht, etc. 1979, pp. 25–40. △

1981 *Philosophical aspects of proof-theory*, in: G. Fløistad and G. H. von Wright (eds.) **Contemporary Philosophy, A New Survey**, vol. **1**, **Philosophy of Language/Philosophical Logic**, Martinus

Nijhoff, The Hague, etc. 1981, pp. 335–277. △

1981a *Validity and normalizability of proofs in 1st and 2nd order classical and intuitionistic logic*, in: S. Bernini (ed.) **Atti del Congresso nazionale di logica**, Montecatini Terme, Italy, October 1–5 1979, Bibliopolis, Naples 1981, pp. 11–36. △

Prawitz, Dag and Per-Erik Malmnäs

1968 *A survey of some connections between classical, intuitionistic and minimal logic*, in: K. Schütte (ed.) **Contributions to Mathematical Logic**, North Holland, Amsterdam, etc. 1968, pp. 215–229. △

Prior, Arthur N.

1962^2 **Formal Logic**, Clarendon Press, Oxford 1962^2. (*Appendix II* contains an informal definition and examples of "condensed detachment", based on [C. A.] Meredith's original explanations.) △

1963 *see* [Meredith *&* Prior 63]. △

de Queiroz, Ruy J. G. B.

1990 **Proof Theory and Computer Programming.** *The Logical Foundations of Computation*, [Ph D Diss., University of London, Imperial College, Department of Computing, February 1990], London 1990, xi + 405 pp. △

1991 *see* [Gabbay *&* de Queiroz 91]. △

1992 *see* [Gabbay *&* de Queiroz 92]. △

1992a *see* [Gabbay *&* de Queiroz 92a]. △

Restall, Greg

2000 **An Introduction to Substructural Logics**, Routledge, London and New York 2000. △ ▽

Rezuş, Adrian

1981 **Lambda-Conversion and Logic**, [Ph D Diss., University of Utrecht 1981], Elinkwijk, Utrecht 1981, ix + 196 pp. △

1982 *On a theorem of Tarski*, **Libertas Mathematica** [Arlington TX] **2**, 1982, pp. 62–95. [Rev. **MR 83c**:03019 and **Zbl 481**.03010.] △

1983 **Abstract Automath**, Mathematisch Centrum [*CWI*], Amsterdam 1983, vi + 188 pp. [ISBN 90-6196-256-0] [*MC Tracts 160*] [Rev. **MR 84j**:03030 and **Zbl 507**.03001.] (Also available online [PDF], since 2004, from `http://www.win.tue.nl/automath/archive/pdf/xaut021.pdf`, *The Automath Archive*, Eindhoven University of Technology.) △ ▽

1983a *see* [Barendregt *&* Rezuş 83]. △

1986 *Semantics of constructive type theory*, **Libertas Mathematica** [Arlington TX] **6**, 1986, pp. 1–82. [Rev. **MR 88a**:03034 and **Zbl 632**.03047.] △

1987 *Propositions-as-types revisited*: *I Higher-order constructive type-theory*, University of Nijmegen, Faculty of Science, Report **KUN-WN TR-97-1987**, February 1987, 91 pp. (No II.) △

1987a *Proof programming*, September 1987, 18 pp. [revised: October 1988]. △

1988 *A type-theoretic approach to classical and non-classical logics*, Talk for a Workshop *Typed Lambda-calculi* held at the University of Nijmegen, November 14–18, 1988. △

1989 *What is a "classical" proof?*, Talk for a *GMD-Kolloquium* held at the *GMD* German National Research Center, *Abt.* Karlsruhe [University of Karlsruhe, *GMD Forschungsstelle für Programmstrukturen*, Computing Department], May 1989 (revised as a technical report: April 1990). △

1990 **Classical Proofs: λ-calculus Methods in Elementary Proof Theory**, August 1990 (also available online from `http://www.equivalences.org`, sub *équivalences / éditions*, 2006). ▽

1991 *Proof semantics for modal logics*: *Glivenko semantics and epistemic models*, June 1991 [revised: October 1991].

1993 **Witness Structures** [in preparation] (cf. also `http://www.equivalences.org`, 2006). △ ▽

van Riemsdijk, H.

1981 *see* [Niekus *et al.* 81].

Saccheri, Gerolamo, SJ (1667–1733)

1697 **Logica demonstrativa**, *quam una cum thesibus ex tota philosophia decerptis, dependendam proponit Joannes Franciscus Caselette...*, Typis Ioannis Baptistae Zappatae, Augustae Taurinorum [i.e., Turin] 1697[1], 8° ([Copy: *Brera* Milano = *Biblioteca Nazionale Braidense*.] Reprint: **L. d.**, *Mit einer Einleitung von Wilhelm Risse. Nachdruck der Ausgabe 1697 Turin.* Olms Verlag, Hildesheim/New York 1983^R, 10 + 12 + 287 pp. There are two other editions: **L. d.,**... *Auctore... Hieronymo Saccherio, S.J....*, Typis Haeredum Caroli Francisci Magrii, Ticini Regii [i.e., Pavia] 1701[2], 167 pp. 8° [copies: **BU** Pavia, **BNC** Rome, **BN** Naples] and **L. d.,**... *Auctore... Hieronymo Saccherio S.J.,...*, Heinrich Noethen, Augustae Ubiorum [i.e., Cologne = Köln] 1735[3], 162 pp., 8° [copies: **UB** Münster, **UStB** Cologne, **BU** Louvain, etc. See also `http://equivalences.org`, sub *équivalences/éditions*, 2006.) ▽

1733 **Euclides ab omni naevo vindicatus;** *sive conatus geometricus quo stabiliuntur prima ipsa universae geometriae principia, auctore Hieronymo Saccherio Societatis Jesu; in Ticinensi Universitate Matheseos Professore Opusculum Ex^{mo} Senatui Mediolanensi ab auctore dicatum...*, Paolo Antonio Montano [typographer], Mediolani [i.e., Milano] 1733, xvi + 142 pp., 4°. ([Copies: **British Museum 8530 e12**, **BN** Paris **V 6060**, etc.] Part I, Latin-English text: **Girolamo Saccheri's Euclides Vindicatus**, "edited and translated by George Bruce Halstead", Open Court, Chicago/London 1920, reprint: Chelsea Publ. Comp., New York NY, 1986^R. See also `http://equivalences.org`, sub *équivalences/éditions*, 2006. There are *partial* translations of Part I: in *German*, by Paul Stäckel and Friedrich Engel, in: **Die Theorie der Parallellinien von Euklid bis auf Gauss**, B. G. Teubner Verlag, Leipzig 1895, pp. 31–135, reprinted by the Johnson Reprint Corporation, New York/London 1968^R, in *Italian*, by Giuseppe Boccardini, in: **L'Euclide emendato del P[adre] Gerolamo Saccheri...,** Ulrico Hoepli, Milano 1904, and in *Interlingue*, by C. E. Sjöstedt, in: C. E. Sjöstedt (ed.) **Le axiome de paralleles de Euclides a Hilbert, un problema cardinal in le evolution del geometrie**. *Excerptes in faccimile... e traduction in le lingue international auxiliari* Interlingue, *Interlingue-Fundation*, Uppsala and *Bokvörlaget Natur och Kultur*, Stockholm 1986, pp. 96-175. Part II [not of direct concern here] has been translated into English by Linda Allegri in her thesis: **The Mathematical Works of Girolamo Saccheri, S.J. (1667–1733)**, [Ph D Diss., Columbia University 1960], 272 pp., University Microfilms, Ann Arbor Michigan 1960 [cf. **Dissertation Abstracts 21** (3), 1960–1961, page 631], order no. **60–03038**.) ▽

Schröder-Heister, Peter J.

1981 **Untersuchungen zur regellogischen Deutung von Aussagenverknüpfungen**, [Ph D Diss. = *Inaugural Dissertation*, University of Bonn 1981], Bonn 1981, 251 pp. △

1982 *Logische Konstanten und Regeln*, **Conceptus 16**, 1982, pp. 45–59. △

1983 *The completeness of intuitionistic logic with respect to a validity concept based on an inversion principle*, **Journal of Philosophical Logic 12**, 1983, pp. 359–377. △

1984 *A natural extension of natural deduction*, **Journal of Symbolic Logic 49**, 1984, pp. 1284–1299. △

1984a *Generalized rules for quantifiers and the completeness of the intuitionistic operators* &, ∨, → , ⊥, ∀, ∃, in: M. M. Richter *et al.* (eds.) **Computation and Proof Theory**, vol. **2**, Springer Verlag, Berlin, etc. 1984 (Proc. Logic Colloquium '83), pp. 399–426 [*Lecture Notes in Mathematics 1104*]. △

1985 *Proof-theoretic validity and the completeness of intuitionistic logic*, in: G. Dorn and P. Weingartner (eds.) **Foundations of Logic and Linguistics: Problems and Their Solutions**, Plenum Press, New York/London 1985, pp. 43-87. △

Schwichtenberg, Helmuth

1996 *see* [Troelstra & Schwichtenberg 96]. △ ▽

1999 **Proof Theory**, Lecture Notes, *Mathematisches Institut der Universität München, Sommersemester 1999*, iv + 76 pp. △ ▽

Scott, Dana S.
1970 *Constructive validity*, in: **Symposium on Automatic Demonstration** (IRIA, Versailles, December 1968), Springer Verlag, Berlin, etc. 1970, pp. 237–275 [*Lecture Notes in Mathematics 125*]. △
Seldin, Jonathan P.
1972 *see* [Curry *et al*. 72]. △
1986 *see* [Hindley & Seldin 86]. △
1987 **Mathesis: The Mathematical Foundation of Ulysses**, Odyssey Research Associates, Ithaca NY & Rome Air Development Center, Air Force Systems Command, Griffiss Air Force Base, New York NY, **RADC TR-87-223 Interim Report**, November 1987, 160 pp. (Unclassified: available "to the general public, including foreign nations" from NTIS, the US National Technical Information Service. Appeared in 1988.) △
1989 *Normalization and excluded middle I*, **Studia Logica 48**, 1989, pp. 193–217.
Sørensen, Morten Heine B. and Peter Urzyczyn
1998 **Lectures on the Curry-Howard Isomorphism**, University of Copenhagen, Department of Computer Science, **DIKU Report 98/14**, May 1998 [x + 261 pp.] (Originally, issued as a set of Lecture Notes for a one-semenster graduate / Ph D course held at DIKU, Copenhagen, 1998–1999. To appear at Elsevier BV, Amsterdam etc., in 2006 [Hardbound, ISBN: 0-444-52077-5].) △ ∇
Stenlund, Sören
1972 **Combinators, λ-terms and Proof Theory**, D. Reidel, Dordrecht 1972. △
van Stigt, Walter P.
1990 **Brouwer's Intuitionism**, North Holland, Amsterdam, etc. 1990 [*Studies in the History and Philosophy of Mathematics 2*]. (Fundamental for the understanding of Brouwer. Therein, *Chapter 1*, pp. 1–19: *The Brouwer bibliography*, "...a complete list of Brouwer's known writings". Contains also copious extracts from Brouwer's *inedita*, with translations into English: *Appendices 1–13*, pp. 387–505.)
Sundholm, Göran
1983 *Constructions, proofs and the meaning of logical constants*, **Journal of Philosophical Logic 12**, 1983, pp. 151–172. △
1983a *Systems of deduction*, in: D. Gabbay and F. Guenthner (eds.) **Handbook of Philosophical Logic 1 Elements of Classical Logic**, D. Reidel [Kluwer Academic Publishers], Dordrecht, etc. 1983, pp. 133–188. △
1986 *Proof theory and meaning*, in: D. Gabbay and F. Guenthner (eds.) **Handbook of Philosophical Logic 3 Alternatives to Classical Logic**, D. Reidel [Kluwer Academic Publishers], Dordrecht, etc. 1986, pp. 471–506. △
Tait, William W.
1967 *Intensional interpretations of functionals of finite type I*, **Journal of Symbolic Logic 32**, 1967, pp. 198–212. (No II.) △
1983 *Against intuitionism: constructive mathematics is part of classical mathematics*, **Journal of Philosophical Logic 12**, 1983, pp. 173–195.
1986 *Truth and proof: the Platonism of mathematics*, **Synthese 69**, 1986, pp. 341–370.
Taylor, Paul
1989 *see* [Girard *et al*. 89]. △
Tennant, Neil
1978 **Natural Logic**, Edinburgh University Press, Edinburgh UK 1978. △
Thiel, Christian
1988 *Die Kontroverse um die intuitionistische Logik vor ihrer Axiomatisierung durch Heyting in Jahre 1930*, **History and Philosophy of Logic 9**, 1988, pp. 67–75. △

Tieszen, Richard L.

1989 **Mathematical Intuition: Phenomenology and Mathematical Knowledge**, Kluwer Academic Publishers, Dordrecht, etc. 1989 [*Synthese Library 203*]. ("[...]A detailed development of a philosophically defensible view [based on Husserl] of the interpretation of the intuitionistic logical constants... This way of looking at the work... results from Heyting's interpretation of the intuitionistic logical constants in terms of a theory of intentionality (in Husserl's sense)..." [*sic*, p. xiii]. Whence, "...a theory of constructions, suitably understood, *is* a theory of mathematical intuition." [p. xii]. Cf. *Chapter 4*, pp. 66–91.) △

Troelstra, Anne S.

1969 **Principles of Intuitionism**, Springer Verlag, Berlin, etc. [*Lecture Notes in Mathematics 95*]. △

1975 *Non-extensional equality*, **Fundamenta Mathematicae 82**, 1975, pp. 307–322. △

1978 *A. Heyting on the formalization of intuitionistic mathematics*, in: E. M. J. Bertin, H. J. M. Bos and A. W. Grootendorst (eds.) **Two Decades of Mathematics in the Netherlands.** *A Retrospection on the Occasion of the Bicentennial of the Wiskundig Genootschap*, (2 vols.), Mathematisch Centrum [*CWI*], Amsterdam 1978, vol. **1**, pp. 153–175. (Excerpts of [Heyting 30], pp. 153-162, with a *Commentary by A. S. Troelstra*, pp. 163-175.)

1980 *The interplay between logic and mathematics: intuitionism*, in: E. Agazzi (ed.) **Modern Logic – A Survey**, D. Reidel, Dordrecht, etc. 1980, pp. 197–221.

1981 *Arend Heyting and his contribution to intuitionism*, **Nieuw Archief voor Wiskunde** (3) **29**, 1981, pp. 1–23.

1981a *see* [Niekus *et al.* 81].

1983 *Logic in the writings of Brouwer and Heyting*, in: V. M. Abrusci *et al.* (eds.) **Atti del Convegno Internazionale di Storia della Logica**, San Gimignano, Italy, December 4–8, 1982, CLUEB, Bologna 1983, pp. 193–210.

1984 *see* [Diller *&* Troelstra 84].

1986 *Strong normalization for typed terms with surjective pairing*, **Notre Dame Journal of Formal Logic 27**, 1986, pp. 547–550.

1988 *see* [Troelstra *&* van Dalen 88].

Troelstra, Anne S. (ed.)

1973 **Metamathematical Investigation of Intuitionistic Arithmetic and Analysis**, Springer Verlag, Berlin, etc. 1973 [*Lecture Notes in Mathematics 344*] (*Corrections and additions*, University of Amsterdam, Department of Mathematics, **Report 74–16**, November 1974. Now also available online `http://turing.wins.uva.nl/ anne/Corr/c344.ps` [17 pp., dated August 1, 2001]. Monograph written by A. S. Troelstra [Chapters I–IV], in collaboration with W. A. Howard, C. A. Smoryński, and J. I. Zucker. "A second corrected edition appeared in 1993 as a report: *ILLC Prepublication Series*, **X-93-05**, *Universiteit van Amsterdam*." [Cf. also `http://turing.wins.uva.nl/ anne/`: the *Bibliography of A. S. Troelstra*.) ▽

Troelstra, Anne S. and Dirk van Dalen

1988 **Constructivism in Mathematics, An Introduction**, (2 vols.), North-Holland, Amsterdam, etc. 1988. (*Corrections* available now online at `http://turing.wins.uva.nl/ anne/Corr/CIM1.ps` [6 pp., dated: April 18, 2000] and `http://turing.wins.uva.nl/ anne/Corr/CIM2.ps` [5 pp., dated: April 28, 2000].) ▽

Troelstra, Anne S. and Helmuth Schwichtenberg

1996 **Basic Proof Theory**, Cambridge UP, Cambridge [UK] 1996. △ ▽

Vailati, Giovanni

1903 *Di un'opera dimenticata del P*[*adre*] *Gerolamo Saccheri* (*Logica Demonstrativa 1697*), **Rivista filosofica 4,** 1903, pp. 528–540. (Reprints: [*separatum*] Pavia 1903, and in: [Vailati 11^R], **CIX,** pp. 477–484.)

1903a *La teoria aristotelica della definizione*, **Rivista di filosofia e scienze affini 5** (2) 1903. (Reprint in: [Vailati 11^R], **CX**, pp. 485–496.)

1904 *A proposito di un passo di Teeteto e di una dimostrazione di Euclide*, **Rivista di filosofia e scienze affini 6** (1) 1904. (Reprint in: [Vailati 11^R], **CXV**, pp. 516–527.)

1904a *Sur une classe remarquable de raisonnements par réduction à l'absurde*, **Revue de métaphysique et de morale 12**, 1904, pp. 804–809. (French translation of [Vailati 04], *con aggiunte*: the additions are also reproduced in a note to the 1911 reprint.)

1911^R **Scritti di G[iovanni] Vailati 1863–1909**, (edited by Mario Calderoni, Umberto Ricci and Giovanni Vacca), B. Seeber [A. Barth], Leipzig [and *Stab. Tip. Aldino*, Florence] 1911. (Cf. also Giovanni Vailati **Scritti**. *A cura di M. Quaranta. Presentazione di Ludovico Geymonat*, 3 vols., Arnaldo Forni Editore, Bologna 1987; [not seen] review by Giorgio Lanaro in: **Rivista di storia della filosofia** (n.s.) **43**, 1988, pp. 812–813.)

Veldman, Wim [Willem Henri Maria]

1987 **Intuitionistische wiskunde** [*Intuitionistic Mathematics*], Lecture Notes, University of Nijmegen, Faculty of Science [Department of Mathematics], 1987, [manuscript] 157 pp. △

de Vrijer, Roel [Roelof Cornelis]

1987 **Surjective Pairing and Strong Normalization: Two Themes in Lambda Calculus**, [Ph D Diss., University of Amsterdam 1987], Eburon, Delft 1987, 129 pp. △

Wansing, Heinrich

1992 *Formulas-as-types for a hierarchy of sublogics of intuitionistic propositional logic*, in: D. Pearce and H. Wansing (eds.) **Nonclassical Logic and Information Processing**, Springer Verlag, Berlin, etc. 1992, pp. 125–145 [*Lecture Notes in Artificial Intelligence 619*]. △

Zucker, Jeffrey

1974 *The correspondence between cut-elimination and normalization I, II*, **Annals of Mathematical Logic 7**, 1974, pp. 1–112, and 113–155. (*Errata* to Part I, *ibid.*, page 156.) △

1977 *Formalization of classical mathematics in Automath*, in: M. Guillaume (ed.) **Colloque international de logique** (Clermont Ferrand, July 18–25, 1975), Editions du Centre National de la Recherche Scientifique, Paris 1977, pp. 135–145.