Review: Ruy J. G. B. de Queiroz *et al.*, The Functional Interpretation of Logical Deduction, World Scientific 2012

Adrian Rezuş

March 6, 2014

Judging by its programmatic title, and on a superficial examination of its table of contents (pp. vii–x), the book under review[1] looks, *prima facie*, as if yet another monograph on the so-called "Curry-Howard Correspondence" (henceforth: CHC – some people use to say, even, "Curry-Howard Isomporhism", casually adding more proper names in the credit-label), also known, following William Howard, as "formulas [or propositions] as types" paradigm.[2] As

[1]Ruy J. G. B. de Queiroz, Anjolina G. de Oliveira, and Dov M. Gabbay, *The Functional Interpretation of Logical Deduction*, World Scientific Publishing Co. Pte. Ltd., New Jersey, London, Singapore etc. 2012 [Advances in Logic 5], xxxii + 266 pp.

[2]We can count about a dozen – or so – of treatises on such topics in print, indeed. Putting aside the rather vast literature on proof- and/or program-verification (Automath, NuPRL, Mizar etc.), on Martin-Löf's constructive type theories and on so-called categorical logic, a shortlist would possibly include the follwing monographs: D. Prawitz *Natural Deduction* (PhD Diss.) Stockholm 1965, (Dover reprint) 2006 (this covers classical, relevant and [Lewis-style] modal logics, as well), S. Stenlund *Combinators, λ-Terms and Proof Theory* Reidel [Springer] 1972, A. S. Troelstra *et al. Metamathematical Investigation of Intuitionistic Arithmetic and Analysis* (LNM 344) Springer 1973, (reprint ILLC, Amsterdam) 1993, G. Helman *Restricted Lambda-abstraction and the Interpretation of Some Non-classical Logics*, PhD Diss. Pittsburgh 1977 (cf. A. R. Anderson, N. D. Belnap Jr. *et al. Entailment 2*, Princeton UP 1992, §71), A. Rezuş *Lambda-Conversion and Logic*, PhD Diss. Utrecht 1981, A. Rezuş *Abstract Automath* (MC [CWI] Tracts 160) CWI Amsterdam 1983, A. Rezuş *Impredicative Type Theories* (originally: Lectures Nijmegen 1985–1986), TR 85/1986, KUN-WNS-Informatica, Nijmegen 1986, J.-Y. Girard *Proof and Types* (originally, a *Cours de DEA*, Paris 7, 1986–1987), in print: Oxford UP 1989, A. S. Troelstra and D. van Dalen *Constructivism in Mathematics* (SLFM 121,123) North Holland [Elsevier] 1988, A. Rezuş *Beyond BHK*, Nijmegen 1991, rev. 1993 (extended abstract in *Dirk van Dalen Festschrift*, Utrecht 1993), H. Barendregt *Lambda Calculi with Types* (in: *Handbook LiCS 2*) Oxford UP 1992, J. R. Hindley *Basic Simple Type Theory* Cambridge UP 1996, reprinted 2002, A. S. Troelstra and H. Schwichtenberg *Basic Proof Theory*, Cambridge UP 1998, (second revised edition) 2000 (passsim), I. Poernomo *et al. Adapting Proofs-as-Programs: The Curry-Howard Protocol* Springer 2005, M. H. Sørensen and P. Urzyczyn *Lectures on the Curry-Howard Isomorphism* (Copenhagen 1998–1999), in print: (SLFM 149) Elsevier 2006, J.-Y. Girard *Le Point Aveugle* (Rome Lectures 2004), in print: Hermann (Paris) 2006–2007 [French], resp. *The Blind Spot* EMS (ETHZ–CH) 2011 [English], H. Barendregt *et al. Lambda Calculus with Types* Cambridge UP and ASL 2013 (mainly Part 1), as well as various lectures, etc. of the reviewer on the subject, largely available since the mid-eighties (for exact references see, e.g., *www.equivalences.org*, under *editions / mathesis*, and, possibly, the survey bibliography on BHK, appended to *Beyond BHK*.)

originally intended (W. Howard *et al.*), the paradigm is, actually, just a formal counterpart of the so-called Brouwer-Heyting-Kolmogorov (BHK) interpretation of intuitionistic logic. Specifically, the subject is usually classified, by current AMS / ASL bibliographic standards, under the rubric (proof-theoretic interpretations of) typed lambda-calculus (resp. typed combinatory logic), while, recently, more philosophically oriented authors would even speak about "proof-theoretical semantics" instead.

This is not exactly the case, on several reasons.

(1) First of all, the book is not a scholarly monograph, by usual standards, but rather a collection of papers published previously. As – honestly – announced in its Preface (p. xi), "the current volume arose out of a sequence of peer-reviewed scientific papers around a non-standard perspective on the so-called Curry-Howard functional interpretation". Actually, with one or two exceptions, we are confronted with verbatim reproductions of material available in print already during the early nineties, and it appears that the authors did not make any effort to update their bibliography. Nevertheless, the book is largely self-contained as it is, and – putting aside the debatable issues noted in part here – makes even a pleasant reading for the newcomer to the subject.

(2) On the other hand, as regards the "non-standard perspective" advertised in the Preface, this one is rather debatable.

(2.1) On the technical side, the main author (Ruy J. G. B. de Queiroz; RdQ, for short) and his main co-worker (actually, one of his former PhD students in Brasil, Anjolina J. de Oliveira; AdO, for short) are advocating a "non-standard" view on CHC as a special case of a vast and more generous approach to logic, due to the third author (Dov M. Gabbay; DMG, for short), known as "Labelled Deductive Systems" (LDS; first installments dated cca 1989; cf., e.g., the Oxford UP 1994 monograph, volume I, bearing the same title).

(2.2) On the philosophical side, the main author (RdQ) attempts to re-evaluate Ludwig Wittgenstein as a (philosophical) forerunner of CHC. (This is, actually, an idea already transpiring in RdQ's London – Imperial College – PhD Diss. 1990, under the supervision of the third author, DMG).

In what follows, I shall mainly focus on the technical – mathematical proper, say, or, rather, logical (here: proof-theoretic) – issues discussed in the book, leaving the philosophical – or other kind of – debate on matters pointed out under (2.2) to would-be Wittgenstein scholars interested in both proof-theory and typed lambda-calculus.

A few introductory remarks are in order. As mentioned before, CHC was meant, initially (for W. Howard, at least, cca 1968), to be a formal – mathematical –, explanation of the idea of a "construction", as applying to (logical) proofs in intuitionism, a formal counterpart of BHK, more or less. In colloquial terms, it should have been a more cautious formulation of the various realizability interpretations of intuitionistic proofs (S. C. Kleene *et al.*).

Prima facie, CHC amounts to the identifications: (1) formula / proposition = type, and (2) rule-of inference = proof-operator. Thus far, this is just a matter of notation and terminology. The lambda-calculus approach adds two more technical analogies: lambda-calculus reductions correspond – along

CHC – to Gentzen-style (natural deduction) détour-eliminations, while lambda-conversions / equalities amount to "proof-isomorphisms" (so that we are ultimately dealing with equational theories). Whence also further parallels, as e.g. confluence [so-called Church-Rosser] + normalisation proofs = proofs of cut-elimination in Gentzen-like sequent systems (a fact first noted by William Tait, during the late sixties). There is also a straightforward combinatory variant of the approach, as well as a category theoretic counterpart of the CHC, applying to intuitionistic proof-systems.

A less convenient thing about (the original) CHC, as regards (first-order) intuitionism at least, consists of the fact the obvious (lambda-calculus) reductions that can be associated to intuitionistic proof-détours are not enough in order to insure confluence [the Church-Rosser property]. Actually, the technique works only for the so-called *Minimalkalkül* of I. Johansson (roughly, intuitionism without negation). For intuitionism – i.e. "Heyting's logic" –, one needs an additional proof-operator (corresponding, more or less, to the Medieval *ex falso quodlibet* [*sequitur*]). In order to obtain a confluent [Church-Rosser] notion of reduction, one needs also many *ad hoc* reduction rules (of the "commuting" and / or of the *ex-falso* kind). On the other hand, the intuitionistic proof-operators should, desirably, satisfy appropriate extensional properties (η-like reduction rules), usually ignored in the traditional proof-theoretic literature (actually, they cannot be expressed formally, in traditional Gentzen terms).

One on the merits of the book under review consists of the fact it addresses the two technical issues referred to in the above. The work is due, mainly, to the second author, AdO, and it was, originally, contained in her PhD Diss. 1995 (cf., mainly, Chapter 4, in the book). Essentially, we have a new reduction system for (first-order) intuitionistic logic, worth looking at separately.[3]

Another novelty of the book consists of the genuine way of accommodating intuitionistic first-order equality in CHC (a matter covered in Chapters 5 and 6); this is a revision of previous ideas due, essentialy, to Per Martin-Löf.

Unfortunately, most considerations meant to apply "beyond intuitionism" are (technically) debatable. This concerns also a departure from recent work on the subject, largely ignored in the book under review.[4]

The main shortcoming of the book – from the point of view of the reviewer – stems from a misunderstanding of the second basic tenet of CHC (as derived from BHK): in intuitionism, rules of inference are to be handled as proof-operators, in this setting, and a proof-operator is associated either to a single introduction-rule or to a single elimination-rule, and conversely (there is a one-one correspondence), thereby defining the provability behaviour of the associated connective. For instance, (intuitionistic) implication is characterised

[3]This is a technical subject for experts in (higher order) term-rewriting systems. Cf., e.g., M. Bezem, J. W. Klop, and R. de Vrijer ["Terese"] (eds.), *Term Rewriting Systems* (Cambridge Tracts TCS 55), Cambridge UP 2003, etc.

[4]Among the authors addressing such issues, in recent times, completely ignored here are M. Felleisen, T. G. Griffin, M. Parigot, M. Sørensen, J. Rehof, P. Urzyczyn, as well as many other authors writing on TCS and/or proof-theoretic topics during the nineties and later, let alone early work done during the eighties, due to the reviewer, and preceeding most of those noted in the above.

– along BHK – by an introduction rule (corresponding to the usual deduction theorem, represented by λ-abstraction in the calculus) and an elimination rule (corresponding to *modus ponens*, represented by functional application). Analogously, (intuitionistic) conjunction has a single introduction rule (the so-called Adjuction rule, corresponding to pair-formation) and two elimination rules (corresponding to a Left and a Right Projection operator, resp.). There is no reason to modify this principle – *idest* CHC as such – for other logics. — Incidentally, this induces a certain "functional" flavour in the understanding of (intuitionistic) proofs. Genuinely classical proofs, however – i.e. those that are not already intuitionistically "correct" –, do not necessarily share the latter feature.

"Classically", if building upon the intuitionistic way of understanding proofs (here: BHK), one needs more proof-operators than in Brouwer-Heyting. Typically, the genuinely classical *reductio ad absurdum*, for instance, corresponds to an additional proof-operator that should be associated to "classical" implication *and* negation (defined inferentially, from implication and a special *falsum* constant). Upon this understanding, the operator corresponding to the intuitionistic *ex-falso* rule becomes a special case of the operator corresponding to *reductio ad absurdum*. Alternatively, if both implication and a primitive *falsum* constant are present in the (classical) logic vocabulary, it is enough to have an operator corresponding to the (intuitionistic) *ex-falso* rule and an additional operator corresponding to the Rule of Clavius (*consequentia mirabilis*). There are many other choices, in fact, since – unlike for intuitionistic logic – the classical connectives are interdefinable. On the other hand, if we want a characterisation of classical implication alone, we need an additional proof-operator corresponding, say, to the Rule of Peirce. (There are, actually, many alternatives to this.) Similar remarks would eventually apply, *mutatis mutandis*, to logics with so-called "classical" features (as, e.g., various relevant and [Lewis-style] modal logics.)

Although feasible, this approach – based, essentially, on BHK – is not necessary for classical logic. (As a matter of fact, BHK was not meant to explain the behaviour of classical proofs.) In other words, a proper understanding of classical proofs need not agree with the original BHK explanations. Specifically, CHC agrees with the BHK plan only on (subsystems of) intuitionistic logic. In general, classical proofs are not functions (in any mathematically sensible meaning of the word); at most, they can be viewed as such only by accident, so to speak.

The way of handling classical logic in the book under review consists, essentially, in an attempt of rescueing the so-called "functional interpretation". This is supposed to be achieved by supplementing the "classical implication"-introduction operator (*idest* the usual λ-abstraction) with an additional introduction rule, together with a restriction on its use, meant to allow a lambda-calculus representation of the so-called Rule of Peirce, without any further additions. The same proof-operator (here: the λ-abstraction) has thus two introduction rules, corresponding resp. to the usual deduction theorem, as in intuitionism, and to a general – otherwise classically invalid – rule, appropriately restricted. Whence, in the absence of additional proof-operators, there is no need for additional reduction resp. conversion rules: classical logic becomes thus, from the authors' point of view, a kind of special case of intuitionism! In particular, the restriction on the second introduction rule is not shown to yield exactly the "witnesses" for classical tautologies (in other words – in view of maximality –, that it does not lead to inconsistency). Moreover, the specific restriction is not preserved under usual [β-] reductions. This makes the proposed extension useless, in a CHC setting.

On the other hand, the "classical" (non-intuitionistic, so to speak) proof-features of the relevant logic connectives[5] are not accounted for, on this plan. There are also well-known additional complications in the attempt to accommodate distributivity principles – characteristic in most relevant logics – in CHC, that are ignored in the book.[6] Similar – and equally trivial – remarks would eventually ruin the strategy of accommodating, on these lines, the (Lewis) modalities in a would-be (extended) lambda-calculus[7], as well as the so-called "linear" logic [LL] proposed by J.-Y. Girard.[8]

In the end, the labelling techniques based on LDSs turn out to be of limited use in a lambda-calculus based account of logical deduction (whether classical, relevant, modal, "linear" or non-intuitionistic in general).

Otherwise, a proof-system for classical logic would correspond (exactly), along a "non-standard" version of CHC (yet on a different labelling resp. "typing" strategy), to the so-called *extended lambda-calculus* (in category-theoretic jargon, *idest* lambda-calculus with surjective pairing, also known as [extensional] $\lambda\pi$-calculus). As most logics deserving the name of a logic and actually occurring in the literature are just subsystems of classical logic[9], one can eventually show that CHC is feasible along an alternative plan, without any reference to the original BHK explanations or else to a would-be "functional interpretation" of the characteristic proof-operators [= rules of inference] therein involved, for that matter.[10]

[5]The ones of system **R** of Anderson & Belnap, for instance. Cf. A. R. Anderson, N. D. Belnap Jr. *et al.* *Entailment 1, 2*, Princeton UP 1975, 1992, especially *2*, §71, due to G. Helman (containing a summary of his Pittsburgh PhD Diss. 1977) and the PhD Diss. of A. Rezuş [Utrecht, 1981], referred to in the above. Actually, Helman handled only (fragments of) relevant logics with (relevant) implication and conjunction in his thesis.The proper way of coping with (relevant) negation in this setting – based on previous work of Dag Prawitz [PhD Diss. Stockholm 1965] and the late Robert K. Meyer [PhD Diss. Pittsburgh 1966] – is due, essentially, to the reviewer [Geneva, cca 1977–1978, unpublished].

[6]In particular, *non-distributive* **R**, also known as **LR** (short for "Lattice **R**") has been studied by R. K. Meyer and some of his collaborators (Canberra ACT) during the mid-eigthies. A λ-calculus-based CHC treatement of first-order **LR** and **R**, relying on previous work due to R. K.Meyer, has been proposed by the reviewer around 1987–1988 [unpublished, Nijmegen; cf. the Bibliography of *Entailment 2*].

[7]Incidentally, a solution for Lewis' **S4** and **S5** is implicit in Dag Prawitz's PhD Diss. [1965], mentioned earlier. (In particular, there is no need for Kripke world-semantics, here.)

[8]For a survey, cf. Girard's Rome 2004 lectures cited before. Girard's LL without "exponentials" [i.e., Lewis **S4**-like modalities] is a *non-distributive relevant logic*, by the way.

[9]Including Lewis-style modal logics, since the Lewis **S5**-modalities are, in fact, "degenerated [first-order monadic] quantifiers", so to speak.

[10]Unpublished work on λ-monoids and "witness structures", due to the reviewer, during the eighties and later.